

The Naive Perspective: Numbers as Symbols and Quantities

Paper I of the Univalent Correspondence Series

YonedaAI Research
Univalent Correspondence Working Group
Independent Research Collective
research@yonedaai.com

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Abstract

This is the first paper in a six-part series, plus synthesis, on what we call the *Univalent Correspondence*: six perspectives on what a natural number “is”. Across the series we proceed from the naive level (this paper), through set-theoretic encoding, universal property, the Yoneda perspective, Homotopy Type Theory (HoTT), and the categorical/structural view, before unifying the latter four under Voevodsky’s univalence axiom. The present paper occupies the *naive* or pre-formal level, where a number is at once a symbol (“58”, LVIII, |||| ||||) and a quantity (the cardinality of a counted collection). We treat numerals as syntax and the underlying number as semantics, and we trace this distinction through the historical arc that runs from Mill’s empiricism, Husserl’s phenomenology of arithmetic, and Frege’s critique of psychologism, into Benacerraf’s identification problem. We argue that the naive level is simultaneously *insufficient*—because of symbol/object confusion, the multiplicity of equally adequate representations, and a foundational underdetermination that already foreshadows the structuralist response—and *indispensable*, because it is the interface where embodied counting, finger arithmetic, and culturally transmitted numeral systems anchor mathematical content in human cognition. We close by showing that the naive level is precisely where the structuralist’s question, “what role does this object play?”, first arises, thereby motivating each of the five formal levels developed in subsequent papers. We also provide a small Haskell artifact demonstrating the syntax-semantics distinction by parsing four numeral systems (Hindu-Arabic, Roman, tally, Babylonian sexagesimal) into a single structural normal form.

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1 Introduction

1.1 The question

What is the number 58? On the page in front of you it is two glyphs, “5” and “8”. In a Roman inscription it is the four-glyph sequence **LVIII**. On a tally sheet kept by a shepherd it is fifty-eight scratches. In Babylonian sexagesimal cuneiform it is a single wedge cluster meaning “58”. Each of these is unambiguously a way of writing 58, and yet they share no graphical features, no spelling, no length. Whatever 58 *is*, it cannot be any one of these inscriptions, for it survives translation between all of them.

This paper is about the level of mathematical reflection at which one notices this fact for the first time. We call this the *naïve* level, in the technical, non-pejorative sense of being prior to formal axiomatization. It is the level of the schoolchild learning to count, of the merchant tallying goods, of the cuneiform scribe, and of every adult human who knows the arithmetical facts of daily life without ever having read a definition of the natural numbers. We will argue that this level is, at the same time, deeply inadequate as a foundation and absolutely indispensable as a phenomenology. The tension between these two claims sets up the entire remainder of the series.

1.2 Place in the series

The series of which this paper is the first instalment is organized around the following six-row table, which we will call the *columns table* throughout:

Level	What 58 is
Naive	A symbol; a quantity
Set-theoretic (von Neumann)	$\{0, 1, \dots, 57\}$
Universal property	The 58th iteration of succ in any initial successor structure
Yoneda	The representable functor $\text{Hom}(-, 58)$
HoTT	A canonical term in the inductive type $\mathbb{N}$, up to path equivalence
Categorical / structural	An invariant under structure-preserving morphisms of arithmetic

Subsequent papers will treat rows two through six in turn, and the synthesis paper will argue that rows three through six are, under the univalence axiom, literally—not metaphorically—the same theory viewed through different windows.

The job of *this* paper is to establish row one with care, both as a positive description of how mathematical content is normally encountered and as a critical diagnosis of why we cannot stop there.

1.3 Outline

Section 3 surveys numeral systems, treating them as syntax, and articulates the syntax–semantics distinction in the form it takes at the naive level. Section 4 develops the dual aspect—numbers as quantities arising from counting and grouping—and connects this to embodied cognition. Section 5 traces the pre-Fregean intellectual history through Mill, Husserl, and Frege, with attention to Frege’s critique of psychologism and his Sinn/Bedeutung distinction. Section 6 states three convergent reasons the naive level is insufficient as a foundation: the symbol/object confusion, multiplicity of representations, and Benacerraf’s identification problem (anticipated already at the naive level). Section 7 argues conversely that the naive level is indispensable: counting, finger arithmetic, and embodied schemas are constitutive of the meaning of arithmetic, not optional pedagogical scaffolding. Section 8 shows that the naive level is the natural locus for the structuralist’s question, “what role does this object play?”, and that this question is what every subsequent paper formalizes. Section 9 states the principal results of the paper as theorems—a syntax/semantics adequacy lemma, a no-canonical-numeral theorem, and a naive Benacerraf principle. Section 10 describes a small Haskell artifact that parses several numeral systems and reduces them all to a single structural normal form. Sections 11 and 12 reflect on connections and future work.

2 Mathematical Framework

This paper is unusual within the series in that its mathematical framework is deliberately minimal: the entire point is to operate *below* the threshold of axiomatic formalization. Nevertheless, we will need just enough formal language to say precisely what numerals are, what counting is, and what it means for two numerals to denote the same number. We collect this machinery here.

2.1 Numerals as words over a finite alphabet

Definition 2.1 (Numeral system). A *numeral system* is a triple $\mathcal{N} = (\Sigma, L, \llbracket \cdot \rrbracket)$ where:

1. Σ is a finite alphabet of *glyphs*;
2. $L \subseteq \Sigma^*$ is a (possibly partial) language of well-formed *numerals*; and
3. $\llbracket \cdot \rrbracket : L \rightarrow \mathbb{N}$ is a *denotation function* mapping each well-formed numeral to a natural number.

We do not, at this stage, presuppose any particular formalization of $\mathbb{N}$; the codomain is the pretheoretic collection of natural numbers as understood by the practitioners of the system.

Example 2.2 (Hindu-Arabic decimal positional system). $\Sigma_{10} = \{0, 1, 2, \dots, 9\}$, L_{10} is the set of finite non-empty strings over Σ_{10} with no leading 0 except for the string “0” itself, and $\llbracket d_{n-1}d_{n-2} \cdots d_1d_0 \rrbracket = \sum_{i=0}^{n-1} d_i \cdot 10^i$.

Example 2.3 (Tally). $\Sigma_{|} = \{|\}$, $L_{|} = \{|\}^*$ (all finite strings), $\llbracket s \rrbracket =$ the length of s . We will sometimes use “gate-five” tally with a closing diagonal stroke; this changes $\Sigma_{|}$ but not the denotation.

Example 2.4 (Roman). $\Sigma_R = \{I, V, X, L, C, D, M\}$ with the usual subtractive constraints determining L_R , and the standard semantics yielding $\llbracket LVIII \rrbracket = 58$.

Example 2.5 (Babylonian sexagesimal). A base-60 positional system using two cuneiform glyphs (one for ones, one for tens) to compose digits 1–59, with positional weights 60^i . The historical absence of a true zero glyph in early Babylonian usage means L is partial and ambiguity is resolved by context; modern transliterations add an explicit zero placeholder.

2.2 Equivalence of numerals

Definition 2.6 (Numeral equivalence). Two numerals $w_1 \in L_1$ and $w_2 \in L_2$ in possibly different numeral systems $\mathcal{N}_1, \mathcal{N}_2$ are *equivalent*, written $w_1 \sim w_2$, iff $\llbracket w_1 \rrbracket_1 = \llbracket w_2 \rrbracket_2$.

Remark 2.7. Already in theorem 2.6, we have implicitly committed to a substantive thesis: that there is some target codomain in which $\llbracket w_1 \rrbracket_1$ and $\llbracket w_2 \rrbracket_2$ can be *compared*. A consistently naive theorist who refuses any abstraction beyond the inscriptions themselves cannot even state when two numerals “mean the same thing”. The bare existence of theorem 2.6 is therefore evidence that the naive level is not internally stable: as soon as one wishes to compare across numeral systems, one is reaching beyond pure syntax.

2.3 Counting as a function

Definition 2.8 (Counting function). Let X be a finite set whose elements are physically or conceptually individuable. A *counting* of X is a bijection $c : \{1, 2, \dots, n\} \rightarrow X$ for some $n \in \mathbb{N}$. The number n is called the *cardinality* of X .

Remark 2.9. The *availability* of theorem 2.8 again presupposes more than the strictly naive theorist allows: it presupposes the natural numbers as an indexing device. We will return to this regress in section 6.

3 Numerals as Syntax

3.1 Positional, additive, and tally systems

The world’s historical numeral systems can be roughly classified along two orthogonal axes: *base* (the size of the radix used to bundle quantities) and *principle* (positional vs. additive vs. pure tally).

Tally. The pure tally is base-1 in the sense that each unit contributes additively and there is no bundling. It is the simplest syntactic system one can imagine. Bone artifacts such as the Lebombo bone (c. 42,000 BCE) and the Ishango bone (c. 18,000–20,000 BCE) are typically interpreted as tallies.

Additive systems. Roman, Egyptian, and Greek alphabetic numerals are additive: distinct glyphs encode distinct values, and the value of a numeral is obtained (with local subtractive corrections) by summing the values of its glyphs. Such systems compress the tally—one glyph for ten, another for fifty, another for a hundred—without yet exploiting position.

Positional systems. The Babylonian sexagesimal system, Mesoamerican vigesimal systems, and the Hindu-Arabic decimal system are positional: a small alphabet of glyphs is reused at each “place,” and the place itself contributes a power of the base. Positional systems are exponentially more compact than additive ones in the length of the numeral required to denote a given number.

Proposition 3.1 (Compression by positionality). *Let $b \geq 2$ be a base. A positional numeral system of base b requires a numeral of length $\Theta(\log_b n)$ to denote n , whereas a pure tally system requires length n , and an additive system with k glyphs of distinct values requires length $\Theta(n/v_{\max})$ where $v_{\max}$ is the largest glyph value (asymptotically linear in n).*

Proof sketch. Standard. For positional, n has $\lfloor \log_b n \rfloor + 1$ digits. For tally, length equals n . For additive, the optimal greedy representation uses $\Theta(n/v_{\max})$ tokens of the largest available glyph, plus $O(1)$ corrections. $\square$

3.2 Syntax versus semantics, naively stated

Definition 3.2 (Naive syntax/semantics distinction). Fix a numeral system $\mathcal{N} = (\Sigma, L, \llbracket \cdot \rrbracket)$. Then:

- the *syntactic* content of a numeral w is w itself, the finite string of glyphs;
- the *semantic* content is the value $\llbracket w \rrbracket$.

The fundamental observation of the naive level is that distinct syntactic strings can have identical semantic content—even *within* one numeral system (e.g. IIII vs. IV in pre-medieval vs. medieval Roman usage)—and *a fortiori* across systems.

We emphasize that theorem 3.2 is a *weak* version of the syntax/semantics distinction. The semantic codomain is the natural numbers treated as a primitive concept; we have not (yet) said what natural numbers are. The remaining papers in the series can each be read as offering a candidate filling of this gap.

3.3 Multiplicity and underdetermination

The fact that there are infinitely many numeral systems for the same underlying $\mathbb{N}$ is not a defect of any one system; it is a feature of the situation. But it generates a diagnostic question that returns again and again throughout the series: *which features of the representation are essential to the number, and which are accidents of the inscription?* Phrasing this question already commits one to thinking of numbers as something *other than* their inscriptions. We will see in section 6 that this is the first crack in the naive edifice.

4 Counting and Quantity

4.1 The dual aspect: number as quantity

The other face of the naive level is the use of numbers to express *quantity*. A flock has 58 sheep; a sentence has 58 letters; a year has 365 days. The number-as-quantity face is older than any numeral system: archeological and ethnographic evidence of one-to-one correspondence between counted items and tally marks predates positional notation by tens of thousands of years.

Definition 4.1 (Naive cardinality, repeated for emphasis). The *cardinality* of a finite collection X is the result of placing the elements of X in one-to-one correspondence with an initial segment of the count sequence. In particular, two collections have the same cardinality iff they can be put into one-to-one correspondence with each other. We do not, at this stage, presuppose anywhere that this number “inhabits.”

Example 4.2 (Hume’s principle, anticipated). The following is sometimes called Hume’s principle (after a remark in Hume’s *Treatise* I.iii.1): the number of F s equals the number of G s iff the F s and the G s can be put in one-to-one correspondence. This is the naive level’s characterization of cardinal identity. Frege famously elevated it to a foundational role in his *Grundlagen* (1884), and neologicism (Wright, Hale, Heck) takes it as definitionally sufficient for arithmetic.

4.2 The count sequence and embodied cognition

The cognitive science literature documents that human and pre-linguistic infant counting relies on at least two distinct systems: a precise *small number system* (subitizing, accurate up to about 4) and an approximate large number system whose Weber-fraction discriminability is roughly logarithmic. Numerals expand the precise system arbitrarily by piggy-backing on language and on culturally transmitted iteration schemes (“next, next, next”). Lakoff and Núñez argue that the count sequence itself is a conceptual metaphor grounded in embodied

schemas: motion along a path, the iteration of grasping motions, the sequential pointing of a finger. We do not endorse all the metaphysics in their account, but we follow them on one thesis that we will need: *the count sequence is not optional*. There is no human acquisition of arithmetic that bypasses the embodied count.

Remark 4.3 (Finger arithmetic). A working adult who has internalized the multiplication table to 9×9 typically retains traces of a finger-based representation, accessible as imagery if not in overt behavior. The Chinese “Chisanbop” technique and the Vedic Indian finger-counting traditions are explicit, mature systems in which finger configurations *are* the numerals. They illustrate that the line between syntax (numerals as inscriptions) and embodied quantity (numbers as counted) is not where one might naively think it is: fingers are inscriptions on the body.

4.3 Counting as proto-induction

Proposition 4.4 (Naive induction principle). *Any property P that (i) holds of an initial segment of the count sequence and (ii) is preserved when one passes from the current count to the next, holds of every count one will reach.*

Discussion. At the naive level, this is not a theorem; it is the implicit principle that governs counting. “Once you’ve counted that high, you can keep going.” Formalizing this principle is the work of the universal-property paper (Paper III) and the HoTT paper (Paper V). What we record here is that the principle is already *operative* at the naive level, in the sense that competent counters use it without knowing they are using it. $\square$

4.4 Quantity does not entail abstract object

A subtle but important observation. The naive use of numbers as quantities does not, by itself, commit one to numbers being objects. “There are 58 sheep” may, on a deflationary reading, mean only that each of the 58 sheep is a sheep and no further sheep is. This deflationary reading is internally consistent and, we believe, captures something real about how numbers function in the most everyday cases. But it cannot survive the move to *arithmetic*—to facts of the form $58 = 2 \cdot 29$ —without some non-trivial reification. We will return to this point in section 6.

5 The Pre-Fregean and Early Fregean Background

5.1 Mill’s empiricism

John Stuart Mill, in his *System of Logic* (1843), Book II, Chapter VI, defended the view that arithmetical propositions are inductive generalizations from physical experience. “ $2 + 1 = 3$ ” is, on his view, a generalization observed to hold whenever one combines a pair of pebbles with a single pebble. This view has the merit of explaining the applicability of arithmetic to the world, but it suffers from at least two well-known defects:

1. Arithmetical propositions appear necessary; inductive generalizations do not. We do not feel that “ $2 + 1 = 3$ ” could fail in the way that “every swan is white” could fail.

2. Arithmetical propositions apply to abstract and even imagined collections in exactly the way they apply to physical collections. The inductive base, on Mill’s view, ought to be physical collections; the extension to abstracta then becomes mysterious.

Frege made the criticism of Mill central in his *Grundlagen* (1884), §§7–10. We side with the orthodox post-Fregean reading on this, while recording that contemporary cognitive science (small-number system, approximate number system, neural representations of magnitude) gives Mill’s account a partial vindication *at the level of psychology*: something *like* an empirical magnitude-sense is part of the acquisition story, even if it is not part of the justification story.

5.2 Husserl’s phenomenology of arithmetic

Edmund Husserl’s *Philosophie der Arithmetik* (1891) attempted to ground arithmetic in mental acts of *collective combination*: the mental act of bringing items together as a multiplicity. Frege’s *Review of Husserl* (1894) is one of the founding documents of analytic philosophy, and its central charge is *psychologism*: the confusion of an objective content (a number, an arithmetical truth) with the subjective act by which a thinker entertains it. Husserl himself later moved away from this view; the mature Husserl (*Logical Investigations*, 1900–01) accepts a sharp ideal/real distinction.

5.3 Frege’s critique of psychologism

Frege’s critique can be summarized as a slogan: *the truth that 7 is prime does not depend on anyone’s thinking it*. If it did, then by varying the thinker one would vary the truth, which is absurd. The same slogan re-purposed against the strict naive level says: *the number 58 cannot be the inscription “58”*, for one and the same number is inscribed by many different glyph sequences.

Remark 5.1 (Sinn and Bedeutung). Frege’s later distinction (*Über Sinn und Bedeutung*, 1892) between *sense* (Sinn) and *reference* (Bedeutung) is the mature articulation of the syntax/semantics distinction we have been developing. “58” and “ $2 \cdot 29$ ” are different senses of the same referent. The naive level treats sense and reference as identical (or at least as undistinguished). Distinguishing them is one of the preconditions of all the formal levels to come.

5.4 Frege’s logicism, briefly

Frege’s positive program in the *Grundlagen* and *Grundgesetze* was to define numbers as extensions of certain second-order concepts: the number of *F*s is the extension of the concept “equinumerous with *F*”. Russell’s paradox showed that the unrestricted comprehension needed for such extensions is inconsistent. This is not the place to recount the recovery program (Russell’s type theory, Zermelo’s set theory, neologicism); we record only that Frege’s specific reduction fails, while his *methodological* thesis—that arithmetic is prior to and independent of psychology—is the orthodox view we take forward into the rest of the series.

6 Why the Naive Level is Insufficient

We give three convergent arguments. They overlap, but each isolates a distinct failure mode.

6.1 The symbol/object confusion

Theorem 6.1 (Symbol/object confusion theorem). *There is no consistent way, working purely at the naive level, to identify a number with its inscription. In particular, if one assumes (i) that the glyph sequence “58” is the number 58, and (ii) that $58 = \text{LVIII}$, then one is forced to identify the glyph sequences “58” and “LVIII”, which are manifestly distinct.*

Proof. By transitivity of identity. If “58” = 58 and $58 = \text{LVIII}$, then “58” = LVIII. But the strings “58” and “LVIII” are distinct strings of distinct glyphs over different alphabets. Contradiction. $\square$

The argument is elementary, but it has weight. It shows that the very *intelligibility* of equations like $58 = \text{LVIII}$ requires distinguishing the number from any one of its inscriptions. The naive level is the level at which this distinction is not yet drawn; the rest of the series is the work of drawing it carefully.

6.2 Multiplicity of representations

Theorem 6.2 (No-canonical-numeral theorem). *There is no privileged numeral system. Specifically, for any finite set of numeral systems $\{\mathcal{N}_1, \dots, \mathcal{N}_k\}$ each adequate for representing every natural number, there is a further numeral system $\mathcal{N}_{k+1}$, not equivalent to any $\mathcal{N}_i$ as a syntactic object, that is also adequate.*

Proof. Given $\mathcal{N}_1, \dots, \mathcal{N}_k$, define $\mathcal{N}_{k+1}$ as the positional system over base $b' = \max_i |\Sigma_i| + 1$. By construction, $|\Sigma_{k+1}| = b'$ exceeds the alphabet size of every $\mathcal{N}_i$, so $\mathcal{N}_{k+1}$ is not equivalent as a syntactic object to any of them. Adequacy is immediate from the standard properties of positional notation. $\square$

Theorem 6.2 is a syntactic version of a more famous theorem we will meet in Paper II:

6.3 Naive Benacerraf

Theorem 6.3 (Naive Benacerraf principle). *Every numeral system $\mathcal{N}_i$ adequate for $\mathbb{N}$ is “equally good” from the standpoint of arithmetic. Hence no $\mathcal{N}_i$ can be the number system; numbers must be the common semantic content shared by all adequate $\mathcal{N}_i$.*

Proof sketch. Let $\mathcal{N}_i, \mathcal{N}_j$ be adequate. For any arithmetical operation $\circ$ definable in both, one verifies that $\llbracket w_1 \circ w_2 \rrbracket_i = \llbracket w'_1 \circ w'_2 \rrbracket_j$ whenever $w_1 \sim w'_1$ and $w_2 \sim w'_2$, where $\sim$ is the cross-system equivalence of theorem 2.6. This witnesses that $\mathcal{N}_i$ and $\mathcal{N}_j$ are interconvertible without arithmetical loss. $\square$

Remark 6.4. This is essentially Benacerraf’s 1965 *identification problem* “one floor down”. Benacerraf’s problem asks why we should regard the von Neumann encoding as canonical when the Zermelo encoding is equally adequate. The naive Benacerraf principle observes the same multiplicity already at the level of numeral systems, before set theory is even brought in. Paper II will pursue Benacerraf at the set-theoretic level proper.

6.4 Convergence: numbers cannot be naive things

The three arguments above (symbol/object confusion, multiplicity of representations, naive Benacerraf) jointly establish:

Theorem 6.5 (Inadequacy of the naive level). *A theory of arithmetic that identifies numbers with their inscriptions, or equivalently with any single privileged numeral system, cannot account for the existence of cross-system equations like $58 = \text{LVIII}$. Therefore, numbers are not naively given things; they are something shared across all adequate inscriptions.*

Proof. Combine theorems 6.1 to 6.3. □

Theorem 6.5 is the negative result of the paper: the naive level cannot be the foundational level. The remainder of the series is the constructive search for what numbers *are*, given that they are not their inscriptions.

7 Why the Naive Level is Indispensable

The negative results of the previous section are sometimes overread, in the foundationalist literature, as showing that the naive level should be *discarded* once the formal levels are in place. We disagree, and this section explains why.

7.1 Indispensability arguments

Proposition 7.1 (Acquisition indispensability). *No human learns arithmetic without learning to count. The count sequence and the elementary numerals are the entry point into the entire mathematical edifice; no formal system of arithmetic has ever been acquired by anyone without first traversing the naive level.*

Empirical sketch. This is an empirical claim, supported by the developmental psychology literature (Piaget, Gelman & Gallistel, Carey). Mathematics education universally proceeds by way of numerals and counting; no curriculum introduces the Peano axioms or the natural number object first. □

Proposition 7.2 (Application indispensability). *Every application of arithmetic to the empirical world proceeds via counting. The bridge from the formal natural numbers to actual sheep, electrons, or seconds is provided by an act of counting (or its extension, measurement), which is a naive operation.*

Discussion. A formalization of arithmetic gives the structural facts (e.g. that $58 = 2 \cdot 29$). It does not, by itself, tell us that there are 58 sheep in the field. The latter requires an act of counting that connects the formal token to the empirical scene. Counting is irreducibly naive in our sense. $\square$

Proposition 7.3 (Pedagogical indispensability). *The introduction of any of the formal levels (universal property, Yoneda, HoTT) presupposes facility with the naive level. The student who is to be taught the universal property of $\mathbb{N}$ must already know what $\mathbb{N}$ is intuitively; the universal property purports to characterize what is being introduced, and the characterization is empty if the introduced object is not already familiar.*

7.2 The naive level as semantic anchor

The cumulative force of theorems 7.1 to 7.3 is that the naive level is what *makes the formal levels semantically non-empty*. A formal system is, considered abstractly, a manipulation of inscriptions according to rules. What gives it mathematical content is the connection to the naive practice it is designed to capture. This connection is not optional; it is constitutive.

Remark 7.4 (Compare: physical theories). This is similar in form to the role played in physics by operational definitions of measurement. A formal theory of mechanics is empty without the bridge laws that say what counts as a measurement of length, of time, of mass. The naive level of arithmetic plays the analogous bridging role between inscription-manipulation and quantitative content.

7.3 Insufficient and indispensable: not a paradox

The two main claims of the paper—that the naive level is insufficient (section 6) and that it is indispensable (section 7)—are not in tension. They occupy different levels of analysis. The naive level is insufficient as a *foundation* because it cannot answer foundational questions; it is indispensable as a *phenomenology* because all foundational work begins from it and returns to it. The whole series should be read as taking this dual stance seriously: the formal levels do real foundational work, but they do so by way of, not by replacement of, the naive level.

8 Bridge to the Formal Levels: The Structuralist’s Question

The diagnostic question that emerges from theorem 6.5 is: if a number is not any of its inscriptions, what *is* it? At the naive level, the natural answer is something like “the number is what all these inscriptions have in common.” Pressed for what they have in common, the naive theorist can offer only: the same place in the count sequence; the same arithmetical role.

Definition 8.1 (The structuralist’s question). The *structuralist’s question*, asked of a putative mathematical object X , is: *What role does X play in the structure that contains it?*

Proposition 8.2 (The structuralist’s question first arises naively). *The structuralist’s question is the natural follow-up to theorem 6.5 as encountered at the naive level. Numbers are what their inscriptions share; what their inscriptions share is the arithmetical role; therefore, numbers are arithmetical roles.*

This formulation is, of course, deeply unsatisfying as it stands. It trades on intuitive notions of “role” and “share” that have not been formalized. The five subsequent papers in the series can be read as five different formalizations of theorem 8.2:

- **Paper II (Set-theoretic).** A number’s role is encoded by its membership profile in a chosen set-theoretic universe. Multiple encodings (von Neumann, Zermelo) instantiate the same role; their underdetermination is Benacerraf’s identification problem.
- **Paper III (Universal property).** A number’s role is its position in the initial successor structure $(\mathbb{N}, 0, \text{succ})$, characterized by its universal property. The underdetermination of Paper II is dissolved.
- **Paper IV (Yoneda).** A number’s role is its representable functor $\text{Hom}(-, n)$. “What role does n play?” becomes “what natural transformations land at n ?”.
- **Paper V (HoTT).** A number’s role is its identity (path) type within the inductive type $\mathbb{N}$. Univalence makes the role-identity equivalence native rather than merely modeled.
- **Paper VI (Categorical / structural).** A number’s role is what is invariant under all structure-preserving morphisms of arithmetic.

The synthesis paper argues that, under univalence, papers III through VI are the same theory.

Remark 8.3 (The naive level is recovered, not erased). A subtle point. The structuralist’s question, in any of its formal guises, gives a precise answer to “what is 58?” But the answer is licensed by the naive level: it is licensed because the structural characterization *recovers* the inscriptions and the counting practice in the right way. A formalization that gave the right structural answer but failed to recover the everyday use of “58” would be deficient as a theory of *arithmetic* (whatever else it might be). The naive level is therefore not only a starting point but a constraint of adequacy on all the formalizations.

9 Results

We collect the principal results of the paper for ease of reference.

Theorem 9.1 (Symbol/object confusion). *The strict identification of a number with any one of its inscriptions is inconsistent with the existence of cross-system equations. (theorem 6.1)*

Theorem 9.2 (No-canonical-numeral). *There is no privileged numeral system; for any finite set of adequate numeral systems there is a further adequate system not equivalent as a syntactic object to any of them. (theorem 6.2)*

Theorem 9.3 (Naive Benacerraf principle). *All adequate numeral systems are interconvertible without arithmetical loss; numbers are the shared semantic content of all of them. (theorem 6.3)*

Theorem 9.4 (Inadequacy of the naive level). *A theory that identifies numbers with their inscriptions, or with any single privileged numeral system, cannot underwrite cross-system arithmetic. (theorem 6.5)*

Proposition 9.5 (Indispensability of the naive level). *The naive level is constitutive of (a) acquisition of arithmetic, (b) application of arithmetic, and (c) pedagogical introduction of formal arithmetic. (Combination of theorems 7.1 to 7.3.)*

Proposition 9.6 (Naive emergence of structuralism). *The structuralist’s question, “what role does n play?”, arises naturally from the inadequacy of the naive level and is the unifying question of papers II through VI. (theorem 8.2)*

10 The Haskell Artifact

To illustrate the syntax/semantics distinction concretely we provide a small Haskell program that parses numerals in four systems—decimal, Roman, tally, and a simplified Babylonian sexagesimal—and reduces them all to a single structural normal form, namely a Peano natural number.

The program is intentionally minimal. Its purpose is not industrial parsing but illustration of the following correspondence: distinct syntactic constructors (the parsers) feed a single semantic constructor (the normal form). What we model in code is exactly the diagram

$$\text{numeral} \xrightarrow{\text{parse}} \text{semantic value (Peano natural)}$$

with several different parsers and one shared codomain. We then test that LVIII, 58, a tally of 58 strokes (i.e. `||||...||||` with 58 vertical bars), and the sexagesimal (58,0) all reduce to the same Peano value.

10.1 Module structure

The artifact is split into the following modules:

- `Naive.Peano`: the structural normal form (Peano naturals) and basic arithmetic.
- `Naive.Decimal`: a Hindu-Arabic decimal parser.
- `Naive.Roman`: a Roman numeral parser handling subtractive notation.
- `Naive.Tally`: a tally parser.
- `Naive.Babylonian`: a simplified base-60 parser.
- `Main`: tests that all four parsers agree on a fixed sample.

The full source is at `src/01-naive/` in the project repository.

10.2 The shared codomain

```
data Peano = Z | S Peano deriving (Eq, Show)

fromInt :: Int -> Peano
fromInt n
  | n <= 0 = Z
  | otherwise = S (fromInt (n - 1))

toInt :: Peano -> Int
toInt Z = 0
toInt (S k) = 1 + toInt k

addP :: Peano -> Peano -> Peano
addP Z y = y
addP (S x) y = S (addP x y)
```

This is the structural normal form into which all four parsers reduce. We will return to this datum in Paper III, where we will recognize **Peano** as an instance of the universal property of $\mathbb{N}$, and in Paper V, where it appears as the inductive type defining $\mathbb{N}$ in HoTT.

10.3 Decimal, Roman, tally, Babylonian

A sketch of each parser:

```
parseDecimal :: String -> Maybe Peano
parseDecimal = fmap fromInt . readMaybe

romanValue :: Char -> Maybe Int
romanValue c = lookup c
  [( 'I', 1), ( 'V', 5), ( 'X', 10), ( 'L', 50), ( 'C', 100), ( 'D', 500), ( 'M', 1000)]

parseRoman :: String -> Maybe Peano
parseRoman = fmap fromInt . roman . map romanValue
  where
    roman vs = case sequence vs of
      Nothing -> Nothing
      Just xs -> Just (foldr step 0 (zip xs (drop 1 xs ++ [0])))
    step (x,y) acc = if x < y then acc - x else acc + x

-- Tally: count only '/'; '/' is a grouping symbol that
-- contributes 0. Any other character fails.
parseTally :: String -> Maybe Peano
parseTally s
  | all ('elem' "|/") s = Just (fromInt (length (filter (=='|') s)))
  | otherwise = Nothing

-- Babylonian: not a string parser. Given a pre-tokenized
```

```

-- list of (digit, place) pairs, return the denotation.
fromBabylonian :: [(Int,Int)] -> Maybe Peano
fromBabylonian places
  | all valid places = Just (fromInt (sum
    [d * 60 ^ p | (d,p) <- places]))
  | otherwise = Nothing
where valid (d, p) = 0 <= d && d < 60 && p >= 0

```

10.4 The agreement test

```

main :: IO ()
main = do
  let dec = parseDecimal "58"
      rom = parseRoman "LVIII"
      tally = parseTally (replicate 58 '|')
      bab = fromBabylonian [(58, 0)]
  putStrLn $ "decimal: " ++ show dec
  putStrLn $ "roman: " ++ show rom
  putStrLn $ "tally: " ++ show tally
  putStrLn $ "babylonian: " ++ show bab
  putStrLn $ "all equal? " ++ show (and
    [ dec == rom, rom == tally, tally == bab ])

```

The point of the test is exactly the point of the paper: four glyphically unrelated inscriptions reduce to the same structural object. The Haskell type system enforces this: all four reduction functions have codomain `Maybe Peano`, so equality of their outputs is meaningful.

Remark 10.1 (Notes on the artifact). Two terminological points are worth flagging. First, `parseTally` counts only `'|'` marks; the diagonal `'/'` is treated as a grouping symbol contributing zero to the count. Hence, on this parser, `"//|||"` and `"|||"` both denote 3. The choice matches the paper’s emphasis on denotation rather than well-formedness of inscription. Second, `fromBabylonian` is not a string parser but a denotation function on pre-tokenized `[(digit, place)]` lists. We retain `parseBabylonian` as a legacy alias in the shipped module. The full source treats this naming explicitly.

11 Discussion

11.1 Connections to the rest of the series

The naive level establishes the desiderata that each formal level must meet. We summarize the form these desiderata take in each subsequent paper:

- *Paper II* (*set-theoretic*) must explain how the multiplicity of inscriptions can be reduced to a single set-theoretic encoding—and must confront the residual underdetermination at the next level (von Neumann vs. Zermelo encoding).

- *Paper III (universal property)* must show that the structural role identified at the naive level is captured by initiality in the category of successor structures; in particular, the unique recursion morphism must reproduce the everyday operations the naive level uses.
- *Paper IV (Yoneda)* must show that the structural role is captured by the representable functor; in particular, the maps from $\text{Hom}(-, n)$ to other functors F correspond to elements of $F(n)$ in a way that respects the everyday operations.
- *Paper V (HoTT)* must show that the structural role is captured by the inductive type, with identity types capturing the intuitive “same number” relation native to the naive level.
- *Paper VI (categorical/structural)* must show that the structural role is exactly the invariant content under structure-preserving morphisms.

In each case, success of the formal level is measured by recovery of the naive practice. This is the methodological role of theorem 8.3.

11.2 The deflationary alternative

A reader might worry that we have stacked the deck in favor of realism about numbers. Could one not, instead, embrace a thoroughly nominalist view in which numerals are all there is, and arithmetic is a syntactic discipline of inscription manipulation? The literature contains serious such proposals (Field, Hellman, in different ways). We do not engage them in detail here; for our purposes it is enough to note that any such program must *still* explain why distinct inscriptions of “58” (decimal, Roman, tally) are interchangeable in arithmetical contexts. That explanation is, in the end, a structural explanation; the nominalist’s parsimony, if successful, is purchased at the cost of accepting structural facts about inscriptions.

11.3 Limitations

We have restricted attention throughout to the natural numbers. The naive level for the rationals, the reals, the complex numbers, the p -adic numbers, and the algebraic and transcendental numbers raises analogous but distinct issues. In particular, the relationship between the naive level and the universal-property level is more delicate for the real numbers, where the naive level is itself partly definitional (“what we mean by a real number is a Cauchy-completion / Dedekind cut / positional infinite expansion”). We defer these to a future companion piece.

11.4 The role of computer science

A subtler limitation: we have spoken of numerals as inscriptions on paper or stone. In the present era much arithmetic happens in machines, where numerals are voltage patterns, register configurations, or floating-point bit fields. The syntax/semantics distinction extends in the obvious way, with the wrinkle that inexact representations (floating-point) raise their

own issues. We mention this only to flag that the naive level is not a fixed historical relic; it is a layer of mathematical practice that travels into new technological substrates without losing its character.

12 Conclusion

We have argued that the naive perspective on number—numerals as syntax, quantities as semantics, with the two related by counting and denotation—is at once foundationally inadequate and practically indispensable. Inadequate, because numbers are not their inscriptions (theorem 6.1), no inscription is canonical (theorem 6.2), and adequate inscriptions are interconvertible (theorem 6.3). Indispensable, because acquisition, application, and pedagogy of arithmetic all proceed through it (theorems 7.1 to 7.3).

The diagnostic move out of the naive level is the structuralist’s question: *what role does n play?* (theorem 8.1). The remaining papers in this series formalize this question in five distinct ways and argue, in the synthesis, that under the univalence axiom the last four formalizations are equivalent.

12.1 Future work

- Extend the naive analysis to rationals, reals, and complex numbers, with attention to the role of completion and continuity.
- Develop the cognitive-science side of the indispensability arguments more thoroughly: small-number system, approximate number system, magnitude representations.
- Pursue the question of inscription on machines (digital, neuromorphic, quantum), where the naive level is mediated by hardware.
- Connect the naive level to category-theoretic “free models”: on a structuralist reading, the naive level corresponds to working with the free model on one generator, before any quotient by structural identifications.

A The Naive Sequence in Detail

For completeness we record the first 60 values of the count sequence in each of our four numeral systems. The point is purely illustrative: the table makes vivid the shared semantic content of glyphically unrelated inscriptions.

Decimal	Roman	Tally (gate-five)	Sexagesimal
0	—	(empty)	0
1	I	—	1
2	II	— —	2
3	III	— — —	3
4	IV	— — — —	4
5	V	— — — — /	5
⋮	⋮	⋮	⋮
10	X	two gate-fives	10
20	XX	four gate-fives	20
50	L	ten gate-fives	50
58	LVIII	11 gate-fives + 3	58
59	LIX	11 gate-fives + 4	59
60	LX	12 gate-fives	1;0

The qualitative jump at row 60 in the sexagesimal column—where the representation acquires a second positional place—illustrates how positional systems amplify compactness at integer powers of the base.

B A Worked Example: Why $58 = \text{LVIII}$ is Non-Trivial

We work the example through in detail, in part to illustrate the mechanics of the syntax/semantics distinction, and in part to show how much of arithmetic the naive theorist already implicitly uses.

Decimal side. The string “58” parses, in the standard positional algorithm, as $5 \cdot 10 + 8$. To compute this we need the multiplication $5 \cdot 10 = 50$ and the addition $50 + 8 = 58$. Both operations are themselves unfolded by the count sequence: we know $50 + 8 = 58$ because counting from 50 by ones for eight steps gives 51, 52, 53, 54, 55, 56, 57, 58.

Roman side. The string LVIII parses, by the additive rule (no leading subtractive in this case), as $\text{L} + \text{V} + \text{I} + \text{I} + \text{I} = 50 + 5 + 1 + 1 + 1$. We compute $50 + 5 = 55$ (counting from 50 by ones for five steps) and $55 + 1 + 1 + 1 = 58$ (three more steps).

Equality. Both sides reduce to 58 (in the count-sequence sense). The equation $58 = \text{LVIII}$ is therefore the assertion that two distinct procedures terminate at the same count. The naive theorist’s grasp of this assertion presupposes (a) the count sequence, (b) the additive principle, (c) confidence that procedures starting from different inscriptions but tracking the same quantity will agree. None of these is a primitive of pure inscription manipulation.

Lifting. The structural lift in subsequent papers will say: forget the procedures; the equation reflects the fact that both inscriptions designate the same element of the initial successor structure / the same Peano term / the same path in $\mathbb{N}$.

C On the Word “Naive”

We have used “naive” throughout in a technical sense; we now record the choice. Two alternative terms are available: *pre-formal* and *pre-theoretic*.

“Pre-formal” is accurate but suggests temporal priority where we intend logical priority: the naive level is not what comes *before* formalization in time but what *lies underneath* it, even after formalization is complete.

“Pre-theoretic” is misleading because the naive level has plenty of implicit theory in it: counting, succession, the additive principle. It is pre-axiomatic, not pre-theoretic.

“Naive” has the disadvantage of pejorative overtones in everyday English; we use it in the established mathematical sense (cf. “naive set theory”, “naive Bayes”) of *prior to formalization but serviceable for many purposes*. We hope the reader will tolerate the term in this technical sense.

References

- [1] S. Awodey, *Structure in Mathematics and Logic: A Categorical Perspective*, *Philosophia Mathematica* **4** (1996), 209–237.
- [2] P. Benacerraf, *What Numbers Could Not Be*, *The Philosophical Review* **74** (1965), 47–73.
- [3] P. Benacerraf, *Mathematical Truth*, *The Journal of Philosophy* **70** (1973), 661–679.
- [4] S. Carey, *The Origin of Concepts*, Oxford University Press, 2009.
- [5] S. Dehaene, *The Number Sense*, Oxford University Press, 1997.
- [6] H. Field, *Science Without Numbers*, Princeton University Press, 1980.
- [7] G. Frege, *Die Grundlagen der Arithmetik*, Breslau, 1884. English translation by J. L. Austin, Blackwell, 1953.
- [8] G. Frege, *Über Sinn und Bedeutung*, *Zeitschrift für Philosophie und philosophische Kritik* **100** (1892), 25–50.
- [9] G. Frege, Review of Husserl’s *Philosophie der Arithmetik*, *Zeitschrift für Philosophie und philosophische Kritik* **103** (1894), 313–332.
- [10] M. Geier (ed.), *Frege’s Lectures on Logic*, Open Court, 2004.
- [11] R. Gelman and C. R. Gallistel, *The Child’s Understanding of Number*, Harvard University Press, 1978.
- [12] B. Hale and C. Wright, *The Reason’s Proper Study*, Oxford University Press, 2001.
- [13] G. Hellman, *Mathematics Without Numbers*, Oxford University Press, 1989.
- [14] E. Husserl, *Philosophie der Arithmetik*, Halle, 1891.

- [15] G. Ifrah, *The Universal History of Numbers*, Wiley, 2000.
- [16] G. Lakoff and R. Núñez, *Where Mathematics Comes From*, Basic Books, 2000.
- [17] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer, 1998.
- [18] J. S. Mill, *A System of Logic*, London, 1843.
- [19] J. Piaget, *The Child's Conception of Number*, Routledge & Kegan Paul, 1952.
- [20] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, 2013. arXiv:1308.0729.
- [21] P. Wadler, *Propositions as Types*, Communications of the ACM **58** (2015), 75–84.
- [22] C. Wright, *Frege's Conception of Numbers as Objects*, Aberdeen University Press, 1983.