

The Universal Property Perspective: Numbers as Initial Successor Structures

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Abstract

We develop the universal-property perspective on the natural numbers and adjacent structures, situating it as the third rung of the six-level ladder of the *Univalent Correspondence*. Following Lawvere’s elementary theory of the category of sets (ETCS), we present the natural numbers object (NNO) $(\mathbb{N}, 0, \text{succ})$ as the initial pointed set with endomorphism: for any pointed dynamical system $(X, x_0, f: X \rightarrow X)$ there exists a unique morphism $h: \mathbb{N} \rightarrow X$ with $h(0) = x_0$ and $h \circ \text{succ} = f \circ h$. From this universal property (a unique map satisfying these two equations) we derive primitive recursion, dependent induction, Lambek’s fixed-point theorem, and the rigidity of $\mathbb{N}$ up to unique isomorphism. We then extend the analysis: π as one half-period of the universal cover $\mathbb{R} \rightarrow S^1$; e as the unique solution to $f' = f$, $f(0) = 1$; $\mathbb{R}$ as the (essentially unique) Dedekind-complete ordered field; $\mathbb{Q}$ as the field of fractions of $\mathbb{Z}$. We dualize through coalgebraic stream types and final coalgebras, contrasting inductive presentations of irrationals (Cauchy sequences modulo equivalence) with coinductive presentations (digit streams as final coalgebras for $X \mapsto D \times X$). The recursion principle and the induction principle emerge as instances of one universal mapping property. We address the structuralist critique: universal properties determine objects only *up to unique isomorphism*, which raises the question *which* isomorphism. Lawvere’s structuralism embraces this, while HoTT (Paper 5) makes it definitional via univalence. We close by sharpening the bridge to Yoneda (Paper 4): a universal property is exactly a representation of a functor; concretely, the NNO is the value $F(\emptyset)$ of the free functor (left adjoint to the underlying-set functor $U: \mathbf{PtEndo} \rightarrow \mathbf{Set}$) at the empty set, and as an initial object it represents the constant singleton functor on the category of pointed endo-systems. Throughout, we accompany the mathematics with a Haskell implementation that realizes the NNO universal property and the recursion combinator generically across endofunctor algebras.

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1 Introduction

1.1 Position in the Six-Level Ladder

The Univalent Correspondence series organizes the question “*what is a number?*” into six levels of abstraction [5]. Paper 01 examined the naive level (numerals as syntax). Paper 02 examined the set-theoretic level (von Neumann ordinals; Benacerraf’s identification problem [3]). The present paper takes the next step. We replace “58 is the set $\{0, 1, \dots, 57\}$ ” with the encoding-independent assertion:

58 is the position obtained by iterating `succ` exactly 58 times from 0 in any initial successor structure.

Two such initial structures are connected by a unique isomorphism, so the position labelled 58 in one corresponds to the position labelled 58 in the other. The number is not a set; it is a node in a universal diagram.

1.2 What Counts as a Universal Property

A *universal property* of an object U in a category $\mathcal{C}$ is a description of U as the unique object (up to unique isomorphism) carrying a specified piece of data such that, for every other object carrying analogous data, there is a unique structure-preserving morphism out of (or into) U [11, 14]. The natural numbers, the real numbers, the field of rationals, the circle,

the exponential function: each admits such a presentation, and each becomes encoding-independent thereby.

The strength of the universal-property perspective is threefold:

1. **Encoding-independence.** The definition refers to morphisms, not elements; the internal constitution of U is opaque.
2. **Compositionality.** Universal morphisms compose as morphisms; structures built by universal property compose accordingly.
3. **Computational content.** The unique morphism guaranteed by the universal property is the recursion (or corecursion) combinator, directly executable as a program.

1.3 The Critique We Will Address

A familiar objection: “up to unique isomorphism” still tolerates many isomorphic copies; choosing one is still a choice. The structuralist response, due to Lawvere and developed by Awodey [1, 2], is that the choice is *semantically irrelevant* because every property of mathematical interest is invariant under isomorphism (*the principle of structuralism*). HoTT will later make this principle *definitional* via univalence (Paper 5). The universal property is the gateway: it teaches us to live with “up to unique iso” as the appropriate identity criterion.

1.4 Outline

Section 2 fixes the categorical framework. Section 3 states and proves the universal property of the NNO and derives recursion and Lambek’s theorem. Section 4 compares the NNO with second-order and first-order Peano arithmetic. Section 5 treats π , e , $\mathbb{R}$, $\mathbb{Q}$ by universal property. Section 6 dualizes to final coalgebras and codata. Section 7 unifies recursion and induction as instances of the universal property. Section 8 addresses the “which iso?” critique. Section 9 bridges to Yoneda. Section 10 discusses the Haskell implementation. Section 11 states our main results, Section 12 discusses, and Section 13 concludes.

2 Mathematical Framework

2.1 Categories, Functors, Pointed Dynamical Systems

Definition 2.1 (Category). A *category* $\mathcal{C}$ consists of a class of objects, for every pair A, B a set $\text{Hom}_{\mathcal{C}}(A, B)$ of morphisms, an associative composition $\circ$, and identities id_A .

Definition 2.2 (Pointed dynamical system). Let $\mathcal{C}$ be a category with a chosen terminal object 1 . A *pointed dynamical system in $\mathcal{C}$* is a triple (X, x_0, f) where $X \in \mathcal{C}$, $x_0: 1 \rightarrow X$, and $f: X \rightarrow X$. A *morphism* of pointed dynamical systems $(X, x_0, f) \rightarrow (Y, y_0, g)$ is a morphism $h: X \rightarrow Y$ in $\mathcal{C}$ such that

$$h \circ x_0 = y_0 \quad \text{and} \quad h \circ f = g \circ h. \tag{1}$$

We denote the resulting category $\mathbf{PtEndo}(\mathcal{C})$ (“pointed endo-systems in $\mathcal{C}$ ”). When $\mathcal{C} = \mathbf{Set}$ we drop the parameter and write $\mathbf{PtEndo}$. The category $\mathbf{PtEndo}(\mathcal{C})$ is canonically isomorphic to the category of algebras for the endofunctor $FX = 1 + X$ in $\mathcal{C}$: the basepoint $x_0: 1 \rightarrow X$ and the dynamics $f: X \rightarrow X$ assemble (via the universal property of the coproduct) into a single algebra map $[x_0, f]: 1 + X \rightarrow X$, and conversely.

The two equations of (1) say exactly that the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{x_0} & X \\ & \searrow y_0 & \downarrow h \\ & & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

commute. The first asserts “ h preserves the basepoint”; the second asserts “ h commutes with the dynamics.”

Definition 2.3 (Initial object). An object I of $\mathcal{C}$ is *initial* if for every $X \in \mathcal{C}$ there is a unique morphism $!_X: I \rightarrow X$.

Lemma 2.4 (Rigidity of initial objects). *If I and I' are initial in $\mathcal{C}$, there is a unique isomorphism $I \xrightarrow{\sim} I'$. In particular, the automorphism group of an initial object is trivial.*

Proof. By initiality of I there is a unique $u: I \rightarrow I'$; by initiality of I' there is a unique $v: I' \rightarrow I$. Both $v \circ u$ and id_I are morphisms $I \rightarrow I$, hence equal by uniqueness; similarly $u \circ v = \text{id}_{I'}$. $\square$

2.2 Endofunctor Algebras and Initial Algebras

Definition 2.5 (F -algebra). Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be an endofunctor. An F -algebra is a pair (X, α) with $\alpha: FX \rightarrow X$. A morphism of F -algebras $(X, \alpha) \rightarrow (Y, \beta)$ is $h: X \rightarrow Y$ with $h \circ \alpha = \beta \circ Fh$. The category of F -algebras is denoted $F\text{-Alg}$.

A pointed dynamical system (X, x_0, f) is precisely an algebra for the endofunctor $FX = 1 + X$ in $\mathbf{Set}$ (or any category with finite coproducts and a terminal object): the algebra map $[x_0, f]: 1 + X \rightarrow X$ packages the basepoint and the dynamics into a single arrow. Algebras for $FX = 1 + X$ are called *successor algebras* or *pointed endo-systems*.

Lemma 2.6 (Lambek). *Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be an endofunctor and let (I, ι) be an initial F -algebra. Then $\iota: FI \rightarrow I$ is an isomorphism.*

Proof. Apply F to obtain $F\iota: FFI \rightarrow FI$. The pair $(FI, F\iota)$ is an F -algebra, hence by initiality there is a unique morphism $u: I \rightarrow FI$ of F -algebras. Then $\iota \circ u: I \rightarrow I$ is an algebra morphism, and so is id_I ; by uniqueness $\iota \circ u = \text{id}_I$. Conversely, since u is by construction an algebra morphism $(I, \iota) \rightarrow (FI, F\iota)$, we have $u \circ \iota = F\iota \circ F u = F(\iota \circ u)$. Substituting the equation $\iota \circ u = \text{id}_I$ just established gives $u \circ \iota = F(\text{id}_I) = \text{id}_{FI}$. Hence ι and u are mutually inverse, and ι is an isomorphism. $\square$

Lambek’s lemma is the categorical analogue of Peano’s third axiom (“**succ** is injective and avoids 0”): the algebra structure $\iota = [0, \text{succ}]: 1 + \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, so 0 and $\text{succ}(n)$ are distinct, and **succ** is injective.

3 The Natural Numbers Object

3.1 Definition and Universal Property

Definition 3.1 (Natural numbers object). A *natural numbers object* (NNO) in a category $\mathcal{C}$ with terminal object 1 is a triple $(N, 0: 1 \rightarrow N, \text{succ}: N \rightarrow N)$ that is initial in $\mathbf{PtEndo}(\mathcal{C})$. Equivalently, $(N, [0, \text{succ}])$ is initial among algebras for the endofunctor $FX = 1 + X$.

Remark 3.2 (Notation for the NNO). We use the symbol N for a generic NNO in an arbitrary category $\mathcal{C}$, and $\mathbb{N}$ specifically when $\mathcal{C} = \mathbf{Set}$ (the standard natural numbers). The two notations refer to the same structural object up to canonical isomorphism (Lemma 2.4); we use $\mathbb{N}$ in the abstract, in informal discussion, and from Section 5 onward, where the working category is $\mathbf{Set}$ (or its enrichment for analysis).

Theorem 3.3 (Universal property of the NNO). *Let $(N, 0, \text{succ})$ be an NNO in $\mathcal{C}$. For every pointed dynamical system (X, x_0, f) in $\mathcal{C}$, there exists a unique morphism $h: N \rightarrow X$ such that*

$$h \circ 0 = x_0 \quad \text{and} \quad h \circ \text{succ} = f \circ h. \quad (2)$$

Proof. This is the definition of initiality in $\mathbf{PtEndo}(\mathcal{C})$ rewritten via Definition 2.2. Existence and uniqueness of h together with the two commuting diagrams are jointly the assertion that (X, x_0, f) is an object of $\mathbf{PtEndo}(\mathcal{C})$ and h is the unique morphism from $(N, 0, \text{succ})$. $\square$

3.2 The Recursion Combinator

The unique map h guaranteed by Theorem 3.3 is the primitive recursion combinator. Writing $\text{rec}(x_0, f) := h$, equation (2) unfolds as

$$\text{rec}(x_0, f)(0) = x_0, \quad \text{rec}(x_0, f)(\text{succ}n) = f(\text{rec}(x_0, f)(n)). \quad (3)$$

These are the recursive equations defining iteration.

Example 3.4 (Addition). Define $\text{add}: \mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$ by $\text{add} := \text{rec}(\text{id}_{\mathbb{N}}, \lambda g. \text{succ} \circ g)$, where the codomain is the function space (assuming $\mathcal{C}$ is cartesian closed). Then $\text{add}(0) = \text{id}_{\mathbb{N}}$ and $\text{add}(\text{succ}n)(m) = \text{succ}(\text{add}(n)(m))$, recovering Peano addition.

Example 3.5 (Multiplication). Define $\text{mul}: \mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$ by

$$\text{mul} := \text{rec}(\text{const}_0, \lambda g. \lambda m. \text{add}(m)(g(m))).$$

The base case $\text{mul}(0)$ is the constant zero function; the step takes a function g representing “multiplication by n ” and returns the function $m \mapsto m + g(m)$, representing “multiplication by $\text{succ}n$.” Each arithmetic operation is a single recursion combinator instance.

Example 3.6 (Position 58). The element $58 \in N$ is $\text{succ}^{58}(0)$. By Theorem 3.3, in any pointed dynamical system (X, x_0, f) , the corresponding “58th iteration” is $\text{rec}(x_0, f)(58) = f^{58}(x_0)$. Two NNOs $(N, 0, \text{succ})$ and $(N', 0', \text{succ}')$ admit unique mutually-inverse morphisms (Lemma 2.4); these match $\text{succ}^{58}(0)$ to $(\text{succ}')^{58}(0')$. The number 58 is exactly this trans-NNO equivalence class.

3.3 The Iteration Principle versus Primitive Recursion

Theorem 3.3 as stated gives the *iteration principle*: from (x_0, f) produce a function $\mathbb{N} \rightarrow X$. Genuine *primitive recursion* allows the recursive call to depend on the index:

$$g(0) = x_0, \quad g(\text{succ}n) = k(n, g(n)). \quad (4)$$

Proposition 3.7 (Primitive recursion from iteration). *In any cartesian category with NNO, primitive recursion as in (4) is derivable from iteration as in (3).*

Proof. Given $x_0 \in X$ and $k: \mathbb{N} \times X \rightarrow X$, set

$$\tilde{x}_0 := (0, x_0), \quad \tilde{f}: \mathbb{N} \times X \rightarrow \mathbb{N} \times X, \quad (n, x) \mapsto (\text{succ}n, k(n, x)).$$

By Theorem 3.3 there is a unique $\tilde{g}: \mathbb{N} \rightarrow \mathbb{N} \times X$ with $\tilde{g}(0) = (0, x_0)$ and $\tilde{g} \circ \text{succ} = \tilde{f} \circ \tilde{g}$. Define $g := \pi_2 \circ \tilde{g}: \mathbb{N} \rightarrow X$. Then $g(0) = x_0$ and $g(\text{succ}n) = k(n, g(n))$. Uniqueness of g follows from uniqueness of $\tilde{g}$ and the fact that $\pi_1 \circ \tilde{g} = \text{id}_{\mathbb{N}}$ by induction. $\square$

3.4 Existence of the NNO

Theorem 3.8 (Lawvere; ETCS [7]). *The category **Set** admits a natural numbers object. Any topos with a natural numbers object satisfies the elementary theory of the category of sets (ETCS) provided one further adds a choice axiom and well-pointedness.*

Sketch. Take $\mathbb{N}$ to be the standard natural numbers (constructed in any background foundation, e.g., as the smallest inductive set; encoding-independent by the universal property). Verify Theorem 3.3 by induction. The choice axiom and well-pointedness are the additional ETCS axioms, see [7, 10]. $\square$

4 Comparison with Peano Arithmetic

4.1 Second-Order Peano Axioms

The second-order Peano axioms are:

P1 0 is a natural number.

P2 For every natural n , $\text{succ}n$ is a natural number.

P3 For every n , $\text{succ}n \neq 0$.

P4 For all m, n , $\text{succ}m = \text{succ}n \Rightarrow m = n$.

P5 (Induction) For every subset $S \subseteq \mathbb{N}$ with $0 \in S$ and S closed under succ , $S = \mathbb{N}$.

Theorem 4.1 (Equivalence with NNO). *A structure $(N, 0, \text{succ})$ in **Set** satisfies P1–P5 if and only if it is a natural numbers object.*

Sketch. ($\Rightarrow$) Suppose $(N, 0, \text{succ})$ is an NNO, i.e., initial in $\mathbf{PtEndo}(\mathbf{Set})$. Since $(N, [0, \text{succ}])$ is the initial algebra for $FX = 1 + X$, Lambek’s lemma (Lemma 2.6) gives that $[0, \text{succ}]: 1 + N \rightarrow N$ is an isomorphism. Reading off the components: 0 and succ have disjoint images (P3, “ $0 \notin \text{im succ}$ ”) and succ is injective (P4). For P5: given a subset $S \subseteq N$ with $0 \in S$ and closed under succ , the inclusion $S \hookrightarrow N$ together with the restricted basepoint and successor makes S a pointed-endo-system; the universal property gives a unique $h: N \rightarrow S$ with $h(0) = 0$ and $h \circ \text{succ}|_S = \text{succ} \circ h$, and the composite $S \hookrightarrow N \xrightarrow{h} S$ together with $\text{id}_N: N \rightarrow N$ both fit into the universal property at target $(N, 0, \text{succ})$, forcing $h = \text{id}_N$ as a function and hence $S = N$.

($\Leftarrow$) Given a target (X, x_0, f) , we construct the unique h in three steps.

Step 1: define a candidate at each level. For each $n \in N$, we wish to produce a partial map $h_{\leq n}: \{0, \dots, n\} \rightarrow X$ satisfying $h_{\leq n}(0) = x_0$ and $h_{\leq n}(\text{succ}k) = f(h_{\leq n}(k))$ whenever $\text{succ}k \leq n$.

Step 2: induct on n to show the family exists. Let

$$S := \{n \in N : h_{\leq n} \text{ exists and is unique on } \{0, \dots, n\}\}.$$

Then $0 \in S$ by setting $h_{\leq 0}(0) := x_0$. Suppose $n \in S$. Define $h_{\leq \text{succ}n}$ on $\{0, \dots, n\}$ to agree with $h_{\leq n}$, and set $h_{\leq \text{succ}n}(\text{succ}n) := f(h_{\leq n}(n))$. This extension is uniquely forced by the recursion equations, so $\text{succ}n \in S$. By P5, $S = N$.

Step 3: patch into a global map. The family $\{h_{\leq n}\}_{n \in N}$ is compatible (each extends the previous), so we may define $h: N \rightarrow X$ by $h(n) := h_{\leq n}(n)$. This h satisfies (2), and uniqueness of each $h_{\leq n}$ on its domain gives uniqueness of h . $\square$

4.2 First-Order PA and Non-Standard Models

First-order Peano arithmetic (PA) replaces the second-order P5 with a schema: induction holds for every formula φ in the language of arithmetic. By the Lowenheim–Skolem theorem PA has non-standard models of every infinite cardinality [15]; these contain “infinite” elements not reachable from 0 by finitely many applications of succ .

Theorem 4.2 (Categorical pinning down). *The standard model is the unique (up to unique isomorphism) NNO in $\mathbf{Set}$. Non-standard models of PA are not NNOs in $\mathbf{Set}$ because they fail the universal property: there exist pointed dynamical systems (X, x_0, f) for which the would-be morphism h is non-unique on the non-standard part.*

The categorical NNO thus offers what first-order axiomatization cannot: a structural criterion that rejects non-standard inhabitants.

5 Other Numbers and Spaces by Universal Property

We now extend the universal-property perspective beyond $\mathbb{N}$.

5.1 The Integers as the Free Group on One Generator

Definition 5.1 (Universal property of $\mathbb{Z}$). $\mathbb{Z}$ is the *free group on one generator*. Concretely: for any group G and any element $g \in G$, there exists a unique group homomorphism $\varphi_g: \mathbb{Z} \rightarrow G$ such that $\varphi_g(1) = g$.

The unique map φ_g is $n \mapsto g^n$. The universal property says equivalently that the forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}_*$ from groups to pointed sets has $\mathbb{Z}$ as the value of its left adjoint at the singleton-pointed set.

5.2 The Rationals as the Field of Fractions

Definition 5.2 (Universal property of $\mathbb{Q}$). $\mathbb{Q}$ together with the canonical embedding $\iota: \mathbb{Z} \rightarrow \mathbb{Q}$ is the *field of fractions* of $\mathbb{Z}$: it is initial among pairs (K, j) where K is a field, $j: \mathbb{Z} \rightarrow K$ a ring homomorphism, and every nonzero element of $\mathbb{Z}$ is sent to a unit. For any such (K, j) there is a unique field homomorphism $\bar{j}: \mathbb{Q} \rightarrow K$ with $\bar{j} \circ \iota = j$.

The unique extension is determined by $\bar{j}(p/q) = j(p)j(q)^{-1}$. The construction (equivalence classes of pairs (p, q) with $q \neq 0$) is one realization of the universal object; the universal property captures the structure encoding-independently.

5.3 The Reals as the Dedekind-Complete Ordered Field

Theorem 5.3 (Universal property of $\mathbb{R}$). *Up to unique order-preserving isomorphism, $\mathbb{R}$ is the unique Dedekind-complete totally ordered field. Equivalently, $\mathbb{R}$ is the terminal object in the category of Archimedean ordered fields with order-preserving embeddings (or initial in the category of ordered fields with an order-preserving Cauchy-completion comparison map).*

Sketch. Existence: any of the standard constructions (Dedekind cuts, Cauchy sequences modulo equivalence). Uniqueness: given two Dedekind-complete ordered fields $\mathbb{R}_1, \mathbb{R}_2$, both contain canonical copies of $\mathbb{Q}$; the order-preserving extension $\mathbb{Q} \rightarrow \mathbb{R}_2$ defined on $\mathbb{R}_1$ via suprema of cuts is forced by completeness, and is bijective by Archimedeanity. $\square$

5.4 The Exponential and the Number e

The exponential function admits multiple universal characterizations.

Theorem 5.4 (Universal property of $\exp$). *The exponential function $\exp: \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is the unique smooth function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

1. $f(0) = 1$;
2. $f'(x) = f(x)$ for all x .

Equivalently, $\exp$ is the unique continuous group homomorphism $(\mathbb{R}, +) \rightarrow (\mathbb{R}_{>0}, \cdot)$ whose derivative at 0 equals 1.

Definition 5.5 (e by universal property). $e := \exp(1)$. Equivalently, e is the unique positive real such that the smooth solution f of $f' = f$, $f(0) = 1$ satisfies $f(1) = e$. By Theorem 5.4, e is well-defined independent of the construction of $\mathbb{R}$.

5.5 The Circle and the Number π

Theorem 5.6 (Universal cover of S^1). *The exponential map*

$$E: \mathbb{R} \rightarrow S^1, \quad E(t) := (\cos t, \sin t),$$

exhibits $\mathbb{R}$ as the universal cover of S^1 . Equivalently, E is the unique continuous group homomorphism $(\mathbb{R}, +) \rightarrow (S^1, \cdot)$ whose derivative at the origin is the unit-speed positively-oriented tangent vector $(0, 1) \in T_{(1,0)}S^1$ (the canonical generator of the Lie algebra $\text{Lie}(S^1) \cong \mathbb{R}$).

Definition 5.7 (π by universal property). π is half the period of E : the smallest positive T with $E(2T) = E(0)$ is $T = \pi$. Equivalently, π is the unique positive real such that $\sin(\pi) = 0$ and $\sin(x) > 0$ for $x \in (0, \pi)$.

Remark 5.8. The circle S^1 can itself be defined by a universal property (as the loop space classifier, or as the higher inductive type with a single point and a single loop). In HoTT (Paper 5) the universal property of S^1 becomes a definitional construction.

6 The Coalgebraic Dual

6.1 Final Coalgebras and Codata

The dual of an initial algebra is a final coalgebra.

Definition 6.1 (F -coalgebra). For an endofunctor $F: \mathcal{C} \rightarrow \mathcal{C}$, an F -coalgebra is a pair (X, γ) with $\gamma: X \rightarrow FX$. A morphism of F -coalgebras $(X, \gamma) \rightarrow (Y, \delta)$ is $h: X \rightarrow Y$ with $\delta \circ h = Fh \circ \gamma$.

Definition 6.2 (Final coalgebra). A *final F -coalgebra* is a terminal object in the category of F -coalgebras: an object $(\nu F, \text{out})$ such that for every F -coalgebra (X, γ) there is a unique morphism $\text{ana}(\gamma): X \rightarrow \nu F$.

The dual of Lambek’s lemma asserts that $\text{out}: \nu F \rightarrow F(\nu F)$ is an isomorphism. The unique morphism $\text{ana}(\gamma)$ is the *anamorphism*, the corecursion combinator.

6.2 Streams as Final Coalgebras

For a fixed alphabet A , set $FX := A \times X$.

Theorem 6.3 (Streams as final coalgebra). *The set $A^{\mathbb{N}}$ of infinite sequences (streams) in A is the final coalgebra of $FX = A \times X$, with $\text{out}(s) := (s_0, \text{tail}(s))$.*

For any $\gamma: X \rightarrow A \times X$, $\text{ana}(\gamma)$ produces the stream of “observations” under repeated application of γ .

6.3 Inductive vs Coinductive Presentations of the Reals

The reals admit at least two HoTT-friendly presentations, corresponding to the two universal-property directions.

Inductive (Cauchy completion). $\mathbb{R}$ is constructed as the quotient of the set of Cauchy sequences $\mathbb{Q}^{\mathbb{N}}$ by the relation $(a_n) \sim (b_n) \iff |a_n - b_n| \rightarrow 0$. This is presented as an initial-algebra-flavoured construction: $\mathbb{R}$ is the smallest set closed under taking limits of Cauchy sequences.

Coinductive (digit streams). For a chosen base $b \geq 2$ and digit set D , the set of b -adic digit streams $D^{\mathbb{N}}$ admits a quotient by “carry” equivalence; the quotient is $[0, 1] \subset \mathbb{R}$. The codomain has a natural *coalgebraic* description: it is the final coalgebra of $X \mapsto D \times X$ (modulo the carry relation as a bisimulation quotient).

Remark 6.4 (Initial vs final tensions in $\mathbb{R}$). The Cauchy and digit-stream constructions are biased toward initial and final structures respectively. They produce isomorphic ordered fields by Theorem 5.3, but the constructions exhibit different *computational* content: Cauchy realizers carry an explicit modulus of convergence; stream realizers carry an explicit observation schedule. In Type II Turing computability, the two are interreducible but not isomorphic as effective structures [18]. We take this as evidence that “ $\mathbb{R}$ ” as a structural invariant lives *above* the choice of inductive vs coinductive presentation.

6.4 Bisimulation and the Coinduction Principle

For codata, equality is captured by *bisimulation*.

Definition 6.5 (Bisimulation). For an F -coalgebra (X, γ) , a *bisimulation* is a relation $R \subseteq X \times X$ such that there exists $\rho: R \rightarrow FR$ making both projections $\pi_1, \pi_2: R \rightarrow X$ coalgebra morphisms. Two elements $x, y \in X$ are *bisimilar* iff $(x, y) \in R$ for some bisimulation R .

Concrete case ($FX = A \times X$, streams). A relation $R \subseteq X \times X$ is a stream-bisimulation iff for every $(x, y) \in R$ with $\gamma(x) = (a_x, x')$ and $\gamma(y) = (a_y, y')$, we have $a_x = a_y$ and $(x', y') \in R$. Two streams are bisimilar iff they are related by some such R , equivalently iff they have equal head and bisimilar tails.

Theorem 6.6 (Coinduction principle). *In the final coalgebra νF , two elements are equal if and only if they are bisimilar.*

This is the coalgebraic analogue of the induction principle (Theorem 7.1 below): just as induction reduces equality of functions out of $\mathbb{N}$ to equality of their basepoint-and-step components, coinduction reduces equality of elements in νF to the existence of a bisimulation.

7 Induction and Recursion as Universal Property Instances

7.1 The Induction Principle

Theorem 7.1 (Dependent induction from the NNO universal property). *Let $(N, 0, \text{succ})$ be an NNO in a locally cartesian closed category $\mathcal{C}$. For every type family $P: N \rightarrow \mathcal{C}$, every*

$p_0 \in P(0)$, and every $p_s: \prod_{n \in N} P(n) \rightarrow P(\text{succ}n)$, there exists a unique dependent function $\text{ind}(p_0, p_s): \prod_{n \in N} P(n)$ satisfying

$$\text{ind}(p_0, p_s)(0) = p_0, \quad \text{ind}(p_0, p_s)(\text{succ}n) = p_s(n, \text{ind}(p_0, p_s)(n)). \quad (5)$$

Proof sketch. Form the total space $E := \Sigma_{n \in N} P(n)$ together with its first projection $\pi: E \rightarrow N$. The pair (p_0, p_s) equips E with the structure of a pointed dynamical system: the base-point is $(0, p_0) \in E$, and the structure map is

$$\tilde{f}: E \rightarrow E, \quad \tilde{f}(n, x) := (\text{succ}n, p_s(n, x)).$$

The projection π is then a morphism of pointed dynamical systems $\pi: E \rightarrow N$ (where N has structure $(0, \text{succ})$). By the NNO universal property (Theorem 3.3), there is a unique morphism $h: N \rightarrow E$ in **PtEndo** with $h(0) = (0, p_0)$ and $h \circ \text{succ} = \tilde{f} \circ h$. The composite $\pi \circ h: N \rightarrow N$ is also a morphism, and is forced by uniqueness to be id_N , so h is a section of π . Define $\text{ind}(p_0, p_s)(n) := \text{snd}(h(n)) \in P(n)$. The defining equations (5) follow from the equations of h . Uniqueness of ind follows from uniqueness of h . $\square$

Remark 7.2. The proposition-as-types correspondence (Paper 01 of the series) interprets a propositional family $P: N \rightarrow \mathbf{Set}$ as a predicate; Theorem 7.1 then reads as:

If $P(0)$ holds, and for every n , $P(n) \Rightarrow P(\text{succ}n)$, then $P(n)$ holds for all n .

This is mathematical induction. The categorical and logical principles are the same theorem.

7.2 Iteration, Primitive Recursion, Course-of-Values Recursion

We have shown that primitive recursion (4) follows from iteration (3) (Proposition 3.7). Course-of-values recursion, where $g(\text{succ}n)$ may depend on all of $g(0), g(1), \dots, g(n)$, is similarly derivable from iteration. We sketch the construction.

Proposition 7.3 (Course-of-values recursion from iteration). *Let $\mathcal{C}$ be cartesian closed with NNO and finite lists. Given $x_0 \in X$ and a step function $k: \text{List}(X) \rightarrow X$ that may consume the entire history, there is a unique $g: N \rightarrow X$ satisfying*

$$g(0) = x_0, \quad g(\text{succ}n) = k([g(0), g(1), \dots, g(n)]).$$

Sketch. Define an auxiliary state space $H := N \times \text{List}(X)$ tracking the index and the history accumulated so far. Set

$$\tilde{x}_0 := (0, [x_0]) \in H, \quad \tilde{f}: H \rightarrow H, \quad (n, \ell) \mapsto (\text{succ}n, \ell ++ [k(\ell)]).$$

By the iteration principle (Theorem 3.3) applied to the pointed dynamical system $(H, \tilde{x}_0, \tilde{f})$, there is a unique morphism $\tilde{g}: N \rightarrow H$ satisfying $\tilde{g}(0) = \tilde{x}_0$ and $\tilde{g} \circ \text{succ} = \tilde{f} \circ \tilde{g}$. Define $g(n) := \text{last}(\pi_2(\tilde{g}(n)))$, the last element of the accumulated history. The defining equations follow by unfolding $\tilde{f}$, and uniqueness of g follows from uniqueness of $\tilde{g}$. $\square$

This pattern — bundling history into a state, iterating, and projecting — is the categorical reading of the standard “strong induction reduces to ordinary induction” move.

7.3 Higher Recursion Schemes

The pattern generalizes. For an endofunctor F admitting an initial algebra μF , the unique morphism $\text{cata}(\alpha): \mu F \rightarrow X$ associated to an F -algebra (X, α) is the *catamorphism* or *fold*. Dually, $\text{ana}(\gamma): X \rightarrow \nu F$ is the anamorphism or unfold. Together they cover the basic recursion schemes of functional programming [13].

8 The Structuralist Critique: Up to Which Isomorphism?

8.1 The Concern

A familiar worry: a universal property determines its object only *up to unique isomorphism*. For numbers we say “ $\mathbb{N}$ is unique up to unique iso,” but in the category **Set** there are many concrete bijective copies of $\mathbb{N}$ (von Neumann ordinals, Zermelo numerals, Church numerals, binary tries). The universal property guarantees a unique iso between any two such copies, but it does not select a particular set as “the” $\mathbb{N}$.

8.2 Lawvere’s Structuralist Response

Lawvere’s position [8, 9], sharpened by Awodey [1, 2], is that this is a *feature, not a bug*:

1. Every property of mathematical interest is invariant under isomorphism.
2. The universal property captures exactly the data needed to specify those properties.
3. Asking “which $\mathbb{N}$ is the real one?” commits a category mistake: one is reifying a label that the structure itself does not record.

This is the *principle of structuralism*: mathematical objects are positions in structures, identified by relational role.

8.3 What HoTT Adds (Forward Reference to Paper 5)

The structuralist response is satisfying philosophically but leaves a gap: in classical mathematics, isomorphism and equality are distinct relations, and one must constantly remember to “identify” isomorphic objects via an explicit transport. HoTT’s univalence axiom [4, 16] closes the gap: it asserts

$$(A = B) \simeq (A \simeq B), \tag{6}$$

making isomorphism and equality definitionally interchangeable.

In the universal-property perspective, this means: “the $\mathbb{N}$ ” is a perfectly good referent, because all candidates are not merely isomorphic but *equal* in the type-theoretic universe. The unique-iso ambiguity dissolves. We will develop this in Paper 5.

8.4 The “Which Iso?” Question Resolved

Even before univalence, the universal property already names a *specific* isomorphism between any two NNOs: the one given by the universal mapping property applied to the second NNO viewed as a pointed dynamical system. This is what is meant by “unique up to *unique* iso.” The deeper question is not just “which iso?” but “is the iso itself unique and structurally determined?” — and the answer is yes. The universal property itself names a single, canonical isomorphism between any two NNOs, with no further choice required: there is no wobble, no ambiguity, no $\text{Aut}(N)$ acting non-trivially on the comparison.

Proposition 8.1 (Canonical iso of NNOs). *Let $(N_1, 0_1, \text{succ}_1)$ and $(N_2, 0_2, \text{succ}_2)$ be NNOs in $\mathcal{C}$. The unique iso $u: N_1 \rightarrow N_2$ provided by Lemma 2.4 is given explicitly by $u = \text{rec}(0_2, \text{succ}_2)$ (the recursion combinator from N_1 targeting N_2), with inverse $\text{rec}(0_1, \text{succ}_1)$. In particular, the automorphism group $\text{Aut}(N) = \{\text{id}_N\}$ is trivial: any pointed-endo-system endomorphism of an NNO is the identity.*

Proof. Apply Theorem 3.3 with target $(N_2, 0_2, \text{succ}_2)$ to obtain u , and again with target $(N_1, 0_1, \text{succ}_1)$ to obtain a candidate inverse v . By Lemma 2.4, u and v are mutually inverse. The triviality of $\text{Aut}(N)$ follows by taking $N_1 = N_2 = N$: both u and id_N are pointed-endo-system endomorphisms of N , hence equal by uniqueness in Theorem 3.3. $\square$

9 Bridge to Yoneda

9.1 Universal Property as Functor Representation

A universal property always exhibits its object as the representing object of some functor. The key task is identifying *which* functor on *which* category, in a way that avoids the tautology of representing $\text{Hom}(N, -)$ by N itself. The correct framing for the NNO uses the fact that initial objects represent constant singleton functors.

9.2 The Free–Forgetful Adjunction

The category **PtEndo** admits a forgetful functor $U: \mathbf{PtEndo} \rightarrow \mathbf{Set}$ sending $(X, x_0, f) \mapsto X$ and remembering only the underlying set. The functor U is part of an adjunction

$$F \dashv U: \mathbf{Set} \rightleftarrows \mathbf{PtEndo},$$

where $F: \mathbf{Set} \rightarrow \mathbf{PtEndo}$ is the *free pointed-endo-system* functor. The existence of F follows from the adjoint functor theorem applied to the locally finitely presentable category **PtEndo**, or constructively from the fact that **PtEndo** is the category of algebras of the finitary endofunctor $1 + (-)$ on **Set**. The defining adjunction isomorphism is, for every set A and every pointed-endo-system (X, x_0, f) :

$$\text{Hom}_{\mathbf{PtEndo}}(F(A), (X, x_0, f)) \cong \text{Hom}_{\mathbf{Set}}(A, U(X, x_0, f)) = \text{Hom}_{\mathbf{Set}}(A, X). \quad (7)$$

We use this isomorphism abstractly; the explicit formula for $F(A)$ is not required for the argument that follows.

Theorem 9.1 (NNO as the free pointed-endo-system on the empty set). *The natural numbers object $(\mathbb{N}, 0, \text{succ})$ is canonically isomorphic to the value of the free functor F at the empty set:*

$$F(\emptyset) \cong (\mathbb{N}, 0, \text{succ}). \quad (8)$$

Equivalently, $\mathbb{N}$ is the initial object of $\mathbf{PtEndo}$. Specializing the adjunction (7) to $A = \emptyset$ recovers the universal property of the NNO: the right-hand side $\text{Hom}_{\mathbf{Set}}(\emptyset, X)$ is the singleton $\{!\}$ (the unique empty function), hence there is a unique morphism $h: \mathbb{N} \rightarrow X$ in $\mathbf{PtEndo}$, which is exactly Theorem 3.3.

Proof. Left adjoints preserve initial objects. The initial object of $\mathbf{Set}$ is $\emptyset$. By the adjunction (7) at $A = \emptyset$, the hom-set $\text{Hom}_{\mathbf{PtEndo}}(F(\emptyset), (X, x_0, f))$ is in natural bijection with the singleton $\text{Hom}_{\mathbf{Set}}(\emptyset, X)$, so $F(\emptyset)$ is initial in $\mathbf{PtEndo}$. By Theorem 3.3 the NNO is also initial in $\mathbf{PtEndo}$, so by Lemma 2.4 there is a unique iso $F(\emptyset) \xrightarrow{\sim} (\mathbb{N}, 0, \text{succ})$, which is the content of (8). $\square$

9.3 Initial Objects Represent the Constant Singleton Functor

The Yoneda content of initiality is that the NNO represents the constant singleton functor.

Corollary 9.2 (Representability of the NNO). *There is a natural isomorphism of functors $\mathbf{PtEndo} \rightarrow \mathbf{Set}$*

$$\text{Hom}_{\mathbf{PtEndo}}((\mathbb{N}, 0, \text{succ}), -) \cong \Delta_1, \quad (9)$$

where $\Delta_1: \mathbf{PtEndo} \rightarrow \mathbf{Set}$ is the constant functor at the singleton 1. By the Yoneda lemma, the universal element of this representation is the unique element of $\Delta_1(\mathbb{N}) = 1$, which is in bijection with $\text{id}_{\mathbb{N}} \in \text{Hom}_{\mathbf{PtEndo}}(\mathbb{N}, \mathbb{N})$.

Proof. By Theorem 9.1, the left-hand side of (9) is, for every (X, x_0, f) , a singleton set; this is exactly $\Delta_1(X, x_0, f) = 1$. Naturality is automatic from Δ_1 being constant. $\square$

Remark 9.3 (The free algebra on the singleton picks out elements). The companion fact, used in many treatments, is that the free pointed-endo-system on the *singleton*, $F(1)$, represents the underlying-set functor $U: \mathbf{PtEndo} \rightarrow \mathbf{Set}$:

$$\text{Hom}_{\mathbf{PtEndo}}(F(1), (X, x_0, f)) \cong \text{Hom}_{\mathbf{Set}}(1, X) \cong X = U(X, x_0, f).$$

The NNO is the special case at $A = \emptyset$. So the same adjunction (7) explains both initiality (representability of Δ_1) and freeness on a single generator (representability of U), simply by varying the input set A .

9.4 Reading the Universal Property Yonetically

The NNO universal property thus has a clean Yoneda interpretation:

1. The free-forgetful adjunction $F \dashv U: \mathbf{Set} \rightleftarrows \mathbf{PtEndo}$ encodes the categorical infrastructure.

2. The NNO is the value $F(\emptyset)$ of the free functor at the empty set: it is the free pointed-endo-system with no extra generators.
3. As an initial object, the NNO represents the constant singleton functor on **PtEndo** (Corollary 9.2).
4. “58” is the position $\text{succ}^{58}(0) \in \mathbb{N}$, which under the unique morphism $\mathbb{N} \rightarrow (X, x_0, f)$ guaranteed by initiality maps to $f^{58}(x_0) \in X$. This position is invariant under all morphisms of **PtEndo** in the precise sense that any morphism $(X, x_0, f) \rightarrow (Y, y_0, g)$ commutes with the unique maps from $\mathbb{N}$.

This recasting illuminates what universal properties are doing in general:

A universal property of an object U in a category $\mathcal{D}$ exhibits U as the value of a left adjoint at a distinguished object of some other category $\mathcal{C}$, equivalently as the representing object for the functor $\text{Hom}_{\mathcal{D}}(c_0, U(-))$ for some $c_0 \in \mathcal{C}$. The Yoneda lemma supplies the universal element, namely the unit of the adjunction at c_0 .

For initial objects, $c_0 = \emptyset$ (in **Set**, or the initial object of $\mathcal{C}$ in general), and the represented functor is the constant singleton. For free objects on a generator, $c_0 = 1$, and the represented functor is the underlying-set functor. We will develop this perspective fully in Paper 4.

9.5 The Slogan

If Paper 02 said “a number is a set,” Paper 03 says “a number is a position in a universal diagram.” Paper 04 will say “a number is a representable functor.” Paper 05 will say “a number is a term up to path equivalence.” The Yoneda perspective is the next refinement, and it agrees with everything we have done.

10 Haskell Implementation

We accompany the mathematics with an executable Haskell library implementing the NNO universal property generically. The full code is in `src/03-universal-property/`; we summarize the design here.

10.1 The Iteration Combinator

The recursion combinator from Theorem 3.3 is the iteration combinator on Peano naturals. We define a generic Peano type, a helper to inject `Int` values into it, and the universal map:

```

1  -- | The natural numbers as the initial successor algebra.
2  data Nat = Z | S Nat deriving (Show, Eq)
3
4  -- | Inject an Int into Peano-style Nat (used for demonstrations
   -- | ).
5  peano :: Int -> Nat

```

```

6 peano n
7   | n <= 0      = Z
8   | otherwise = S (peano (n - 1))
9
10 -- / The universal property: unique map from Nat to any (X, x0,
11    f).
12 --   nnoRec x0 f n = f^n x0
13 nnoRec :: x -> (x -> x) -> Nat -> x
14 nnoRec x0 _ Z      = x0
15 nnoRec x0 f (S n) = f (nnoRec x0 f n)

```

The two equations of `nnoRec` are exactly the two equations (3).

10.2 Generic Initial Algebras

We then implement the `fold/cata` combinator for arbitrary initial algebras:

```

1 newtype Mu f = Mu { unMu :: f (Mu f) }
2
3 cata :: Functor f => (f a -> a) -> Mu f -> a
4 cata alpha = alpha . fmap (cata alpha) . unMu

```

The Peano case is recovered by setting $FX = 1 + X$:

```

1 data NatF a = ZeroF | SuccF a deriving Functor
2
3 type GenNat = Mu NatF
4
5 zeroG :: GenNat
6 zeroG = Mu ZeroF
7
8 succG :: GenNat -> GenNat
9 succG n = Mu (SuccF n)
10
11 -- / Generic recursion, instance of the universal property.
12 genRec :: x -> (x -> x) -> GenNat -> x
13 genRec x0 f = cata (\case ZeroF -> x0; SuccF y -> f y)

```

10.3 Final Coalgebras and Streams

For codata we dualize:

```

1 newtype Nu f = Nu { unNu :: f (Nu f) }
2
3 ana :: Functor f => (a -> f a) -> a -> Nu f
4 ana gamma = Nu . fmap (ana gamma) . gamma
5
6 data StreamF a x = Cons a x deriving Functor

```

```

7
8 type Stream a = Nu (StreamF a)
9
10 streamHead :: Stream a -> a
11 streamHead s = case unNu s of Cons x _ -> x
12
13 streamTail :: Stream a -> Stream a
14 streamTail s = case unNu s of Cons _ t -> t

```

10.4 Demonstration: 58 in Multiple Successor Structures

The main demonstration verifies Example 3.6: “58” computes the same trajectory in different pointed dynamical systems, exemplifying that 58 is a structural position rather than a particular value.

```

1 -- 58 in (Int, 0, +1)
2 demo1 :: Int
3 demo1 = nnoRec 0 (+1) (peano 58)
4
5 -- 58 in (String, "", ('*' :))
6 demo2 :: String
7 demo2 = nnoRec "" ('*' :) (peano 58)
8
9 -- 58 in ([Int], [], step) where step prepends the current list
   length
10 demo3 :: [Int]
11 demo3 = nnoRec [] (\xs -> length xs : xs) (peano 58)

```

The number 58 is the unique morphism’s value at the position $\text{succ}^{58}(0)$; that is what the function `nnoRec` computes, and it commutes with every choice of pointed dynamical system.

11 Results

We collect the main results of the paper.

Theorem 11.1 (Main result — structural identity of $\mathbb{N}$). *The natural numbers are determined up to unique isomorphism by the NNO universal property (Theorem 3.3). The notion “58” admits an encoding-independent meaning as the unique image of $\text{succ}^{58}(0)$ under the canonical map into any pointed dynamical system. Two NNOs are connected by a unique iso (Lemma 2.4, Proposition 8.1); their copies of 58 correspond.*

Theorem 11.2 (Main result — universal-property catalogue). *The constants π , e , the structures $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, S^1 , and the functions $\exp$, $\sin$, $\cos$ each admit a universal property in an appropriate category. These universal properties are encoding-independent and refer only to the categorical role of each object.*

Theorem 11.3 (Main result — inductive/coinductive duality). *The recursion principle (initial algebra) and the corecursion principle (final coalgebra) are dual instances of universal properties. Inductive presentations (e.g., Cauchy reals) and coinductive presentations (e.g., digit-stream reals) of the same structural object yield isomorphic structures by the universal property; the inductive/coinductive choice is a presentational rather than ontological matter.*

Theorem 11.4 (Main result — representability). *A universal property exhibits its object as the value of a left adjoint at a distinguished generator, equivalently as a representing object for a functor describable without reference to itself. For the NNO: $\mathbb{N} = F(\emptyset)$ is the value of the free functor $F: \mathbf{Set} \rightarrow \mathbf{PtEndo}$ (left adjoint to the forgetful U) at the empty set, and as an initial object it represents the constant singleton functor on $\mathbf{PtEndo}$ (Theorem 9.1, Corollary 9.2). This is the bridge from the universal-property perspective (this paper) to the Yoneda perspective (Paper 4).*

12 Discussion

12.1 Universal Properties as a Foundational Style

The universal-property style of definition has proliferated through 20th- and 21st-century mathematics: limits and colimits, free objects, adjoint functors, quotient constructions, classifying spaces. Each universal property packages an existence-and-uniqueness statement and, in cases where the relevant category is small enough, computational content (the unique morphism is constructively producible).

The universal-property perspective is more parsimonious than the set-theoretic perspective in two ways. First, it does not require choosing an encoding. Second, it scales: “define X ” becomes “state the universal property of X ,” which automatically transfers across categories whenever the universal property is satisfiable.

12.2 Limitations

Three limitations are worth noting:

Existence is not free. The universal property of the NNO does not by itself guarantee that an NNO exists; one must verify it in the working category. ETCS adds existence as an axiom; topos theory verifies existence in many natural toposes [7, 12].

Choice and impredicativity. Some universal properties (e.g., the universal cover of a space) require background choice principles or impredicative quantification. The natural numbers do not, but the reals (via Dedekind completeness) implicitly assume powerset-like operations.

The which-iso question persists in classical mathematics. As discussed in Section 8, classical mathematics maintains the distinction between isomorphism and equality. The universal property names a unique iso, but this remains a separate datum from equality. HoTT removes the distinction.

12.3 Connections to Computation

The recursion combinator rec from Theorem 3.3 is the primitive recursion combinator of Gödel’s System T. The catamorphism on μF is the fold of functional programming. The anamorphism on νF is the unfold. Hylomorphisms (compositions $\text{cata} \circ \text{ana}$) capture divide-and-conquer recursion schemes. These computational artifacts are not coincidentally analogous to the categorical concepts; they are exactly the categorical concepts viewed under the lens of the Curry–Howard–Lambek correspondence [6, 17].

12.4 Connections to Other Levels of the Series

To Paper 01 (Naive level). The universal property dissolves the naive question “what *is* the symbol 58?” into the structural question “what does 58 *do* in any successor structure?”

To Paper 02 (Set-theoretic level). The universal property explains why Benacerraf’s identification problem is not a problem: every encoding satisfying the universal property is canonically interconvertible with every other.

To Paper 04 (Yoneda). The universal property is the rough-cut form of which Yoneda is the systematic theory: every universal property is a representation, and every representation can be read as a universal property.

To Paper 05 (HoTT). HoTT internalizes the universal property in two ways: the inductive type $\mathbf{Nat}$ satisfies the NNO universal property by its elimination rule, and the univalence axiom upgrades “up to unique iso” to “definitionally equal.”

To Paper 06 (Categorical/structural level). The universal property is the formal vehicle for the structuralist thesis. “58 is a structural invariant” means precisely “the universal property of $\mathbf{N}$ identifies a position $\text{succ}^{58}(0)$ that is invariant across initial successor structures.”

13 Conclusion

We have presented the universal-property perspective on numbers, with the natural numbers object as the central case and $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \pi, e, S^1$ as further instances. The universal property of the NNO is captured in a single equation pair (Theorem 3.3); from it follow recursion (Proposition 3.7), induction (Theorem 7.1), Lambek’s lemma (Lemma 2.6), the rigidity of the natural numbers (Lemma 2.4, Proposition 8.1), and the structural meaning of position 58 (Example 3.6, Theorem 11.1).

The coalgebraic dual provides codata: streams, infinite trees, and coinductive presentations of the reals. The two directions are complementary, not competing; the universal-property perspective regards them as inductive/coinductive presentations of one structural invariant.

Two open questions for the remaining papers in the series:

1. How does the adjunction $F \dashv U$ of Theorem 9.1 interact with the Yoneda embedding $\mathbf{PtEndo} \hookrightarrow [\mathbf{PtEndo}^{\mathrm{op}}, \mathbf{Set}]$, and how is the constant singleton functor Δ_1 recovered from the representable presheaf $\mathrm{Hom}_{\mathbf{PtEndo}}(\mathbb{N}, -)$? (Paper 4.)
2. How do higher inductive types in HoTT subsume universal properties for spaces (e.g., S^1 as a HIT with one point and one loop)? (Paper 5.)

The universal-property perspective has prepared the ground: numbers are positions in initial structures; everything else is encoding. Paper 4 will recast this in the language of representable functors, and Paper 5 will internalize it as definitional equality. The destination is univalent foundations; this paper is one rung of the ladder.

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