

The Univalent Correspondence: How Six Perspectives on Number Become One

Synthesis of the Univalent Correspondence Series

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Abstract

This is the synthesis paper for the six-part series *The Univalent Correspondence*. Each topical paper takes one perspective on the question “what is the natural number 58?” and develops it to maturity: Paper I treats 58 naively as a symbol or quantity; Paper II encodes it set-theoretically as a von Neumann ordinal; Paper III characterizes it by a universal property as the 58th iterate of `succ` in any initial successor structure; Paper IV identifies it with its representable functor $\text{Hom}(-, 58)$; Paper V realises it as a canonical term of an inductive type $\mathbb{N}$ in Homotopy Type Theory, identified up to path equivalence; Paper VI reads it structurally, as an invariant under all structure-preserving morphisms between models of arithmetic. We argue that under Voevodsky’s univalence axiom, the four perspectives of Papers III through VI are not analogous but *literally identical*: an initial successor structure, a representing presheaf, a contractible Σ -type of NNO structures, and a structural invariant of arithmetic models are four faces of one ∞ -groupoid. We name this identification the *univalent correspondence*, building on the four-way computational tetrad of Curry, Howard, Lambek, and Voevodsky (extending Wadler’s propositions-as-types-as-spaces and Harper’s computational trinity). After reviewing each level, motivating the canonical six-row table, and making the unifying ∞ -groupoid framework precise, we prove transit theorems showing that each level transcends its predecessor: the set-theoretic level resolves the naive symbol/object confusion but introduces encoding-relativity (Benacerraf’s identification problem); the universal property resolves encoding by going relational; Yoneda makes “relational profile” mathematically precise as a representable functor; HoTT makes Yoneda’s “iff” into a literal identity in the universe via univalence; and the categorical/structural perspective is the philosophical commitment that *all of this* is what mathematical objects are. We close with a research outlook on transcendentals π and e in HoTT via Cauchy reals as a higher inductive-inductive type, directed HoTT and synthetic ∞ -category theory, and the principal open problem: a synthetic foundation for analytic number theory.

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1 Introduction

1.1 The question and the six perspectives

The Univalent Correspondence series asks one question — *what is the natural number 58?* — and answers it six different ways. The six papers of the series each occupy one row of table 1, the central organizing artifact of the project. The synthesis paper before you argues that the rightmost four rows describe the *same* mathematical object, identified up to equivalence by Voevodsky’s univalence axiom.

Level (Paper)	What 58 is at this level
Naive (I)	A symbol on the page; a quantity counted by a finger or scratch.
Set-theoretic (II)	The von Neumann ordinal $\{0, 1, \dots, 57\}$; equivalently, any iso-class of ZFC encodings of the same Peano structure.
Universal property (III)	The 58-th iterate $\text{succ}^{58}(\text{zero})$ in <i>any</i> initial successor structure $(N, 0, \text{succ})$ in any category with terminal object.
Yoneda (IV)	The representable functor $\text{Hom}(-, 58)$ on the appropriate category; equivalently, the entire system of probes that land on 58.
HoTT (V)	A canonical term in the inductive type $\mathbb{N}$, identified up to path equivalence (and hence, by univalence, identified across NNO presentations).
Categorical / structural (VI)	A semantic invariant, in the sense of Awodey, Resnik, and Shapiro, under all structure-preserving morphisms between models of arithmetic.

Table 1: The six perspectives on the natural number 58. Rows III–VI become literally identical under the univalence axiom (section 3).

The series is not a survey. Each topical paper is a self-contained mathematical development with its own theorems, proofs, and Haskell artifacts. Paper I [1] is historical and philosophical, anchoring the discussion in numerals and counting; Paper II [2] axiomatizes

ZFC, develops the von Neumann hierarchy, and proves Benacerraf’s identification problem in formal terms; Paper III [3] introduces the natural numbers object, derives primitive recursion from its universal property, and shows the equivalence with second-order Peano arithmetic; Paper IV [4] proves the Yoneda lemma in covariant and contravariant forms, with applications to numbers, groups, schemes, and the ∞ -categorical generalization; Paper V [5] is a self-contained development of HoTT including identity types, path induction, the univalence axiom, higher inductive types, and the cubical realisation; Paper VI [6] surveys mathematical structuralism, proves an isomorphism-invariance theorem for first-order properties, and relates categorification to decategorification.

The synthesis paper before you assumes the existence and content of these six topical papers and combines them. We do not re-prove their main theorems; we cite them. What we do prove are the *transit theorems* between levels, and we collect them into a single mathematical statement of the univalent correspondence (sections 3 and 5).

1.2 The thesis

The thesis of this paper, and of the series as a whole, has two layers.

Principle 1.1 (Univalent Correspondence, weak form). For the natural numbers, and more generally for any mathematical structure presentable by an essentially algebraic theory, the four perspectives *universal property* (Paper III), *Yoneda* (Paper IV), *HoTT term up to path equivalence* (Paper V), and *structural invariant* (Paper VI) yield equivalent characterisations.

Principle 1.2 (Univalent Correspondence, strong form). Under the univalence axiom, the equivalences of theorem 1.1 are internalised as identities in a universe $\mathcal{U}$. The four characterisations do not merely yield isomorphic answers; they are paths in the same ∞ -groupoid, and the natural number 58 is a single point in that ∞ -groupoid, viewed through four windows.

The strong form is the content of theorem 4.2 below. The weak form is established meta-theoretically by the work of Lawvere [34], Lambek [27], Wadler [26], Awodey [59], and the HoTT Book [7]. The strong form requires univalence and the structure identity principle (SIP); see section 3.5.

1.3 Why a sixth row?

A natural question: if the universal property already characterises 58 up to canonical isomorphism, why do we need Yoneda, HoTT, and the structural view as separate rows? The short answer: each row resolves a residual question raised by the previous one.

- **Naive (I) $\rightarrow$ Set-theoretic (II).** The naive level confuses the symbol 58 with the number 58. The set-theoretic level resolves the confusion by exhibiting a definite mathematical object $\{0, 1, \dots, 57\}$.
- **Set-theoretic (II) $\rightarrow$ Universal property (III).** The set-theoretic encoding is non-canonical (Benacerraf): von Neumann’s and Zermelo’s encodings disagree about whether $\{\emptyset\} \in 17$. The universal property abstracts away from the encoding by characterising the structure up to canonical isomorphism.

- **Universal property (III) $\rightarrow$ Yoneda (IV).** Universal properties give a relational identification one structure at a time. The Yoneda lemma generalises this to a universal principle: *every* object is determined by its system of incoming morphisms. “58 is initial-among- X ” becomes “58 *is* the functor $\text{Hom}(-, 58)$.”
- **Yoneda (IV) $\rightarrow$ HoTT (V).** The Yoneda lemma yields $X \cong Y \iff h_X \cong h_Y$ as a meta-theorem about a category. Under univalence, this “iff” becomes a literal equality in the universe: the implication $h_X = h_Y \rightarrow X = Y$ is constructed by transport, and the implication $X \cong Y \rightarrow X = Y$ is the univalence axiom.
- **HoTT (V) $\rightarrow$ Categorical/Structural (VI).** HoTT is a formal system. The categorical/structural view is the philosophical commitment that *this is what mathematical objects are*: positions in structures, identified up to equivalence-respecting morphisms. SIP makes this philosophical commitment provable inside HoTT.

1.4 Naming the correspondence

The four rightmost rows of table 1 correspond to the four columns of the *computational tetrad*, the four-way extension of the Curry–Howard–Lambek correspondence shown in table 2. We follow Wadler [26] in calling its central insight *propositions-as-types-as-spaces*, but we call its strong form (internalised by the univalence axiom) *the univalent correspondence*. The terminology and its history are surveyed in section 7.

Logic	Type theory	Category theory	Homotopy topology	/
Propositions	Types	Objects	Spaces	
Proofs	Terms	Morphisms	Points / paths	
Implication $A \rightarrow B$	Function type $A \rightarrow B$	Hom-set	Path space	
Conjunction $A \wedge B$	Product $A \times B$	Product / limit	—	
Disjunction $A \vee B$	Sum $A + B$	Coproduct / colimit	—	
Truth $\top$	Unit $\mathbf{1}$	Terminal object	Point	
Falsehood $\perp$	Empty $\mathbf{0}$	Initial object	Empty space	
Equality $a = b$	Identity $\text{Id}_A(a, b)$	Isomorphism	Homotopy equivalence	
Universal $\forall$	Π -type	Right adjoint / limit	Fibration	
Existential $\exists$	Σ -type	Left adjoint / colimit	Total space	

Table 2: The computational tetrad. Each row is a single concept seen through four windows. Adapted from [26, 7]; the explicit fourth column is due to Voevodsky’s univalent foundations project.

1.5 Outline

Section 2 reviews the six perspectives row by row, summarising the principal results of Papers I through VI. Section 3 proves the five transit theorems that move from one level to the next. Section 4 sets up the unified mathematical framework: an ∞ -groupoid model under univalence, in which the four rightmost perspectives correspond to four functors out of one ∞ -topos. Section 5 states and proves the main identity-of-perspectives theorem. Section 6 makes the inter-level commutativity precise via commutative diagrams of equivalences. Section 7 surveys the history of the naming and explains our preference for “the univalent correspondence.” Section 8 sketches two downstream applications: transcendentals π and e in HoTT and directed HoTT for representation theory. Section 9 collects open problems. Section 10 concludes.

2 Row by Row: The Six Perspectives in Brief

This section is a synopsis of the six topical papers. We assume the reader has access to the topical papers and reproduce only enough material to make the synthesis self-contained.

2.1 Paper I: Naive — numbers as symbols and quantities

Paper I [1] occupies the pre-formal level. A natural number 58 is, at this level, a symbol on a page (“58”), a Roman numeral (LVIII), a tally (fifty-eight scratches), or a Babylonian sexagesimal cluster, and simultaneously a quantity (the cardinality of a counted collection). The paper traces this dual aspect through Mill’s empiricism, Husserl’s phenomenology of arithmetic, Frege’s critique of psychologism in *Grundlagen der Arithmetik* [49] and *Sinn und Bedeutung* [50], and Benacerraf’s identification problem [47].

The principal lesson of Paper I is the syntax/semantics distinction: numerals are syntactic objects (symbols, words over an alphabet), while numbers are the semantic invariants that all adequate numeral systems share. The paper provides a Haskell artifact parsing four numeral systems (Hindu–Arabic, Roman, tally, Babylonian sexagesimal) into a single structural normal form.

The naive level is shown to be simultaneously *insufficient* (because of symbol/object confusion, multiplicity of equally adequate representations, and a foundational underdetermination that already foreshadows the structuralist response) and *indispensable* (because embodied counting and culturally-transmitted numerals anchor mathematical content in human cognition). The paper closes by motivating each of the five formal levels developed in subsequent papers.

2.2 Paper II: Set-theoretic — numbers as von Neumann ordinals

Paper II [2] works in Zermelo–Fraenkel set theory with Choice (ZFC). It develops the von Neumann encoding $0 = \emptyset$, $n + 1 = n \cup \{n\}$, in which $58 = \{0, 1, \dots, 57\}$, and contrasts it with Zermelo’s encoding $0 = \emptyset$, $n + 1 = \{n\}$. Both yield structures isomorphic to $(\mathbb{N}, 0, \text{succ})$ as Peano models, yet they differ on internal set-theoretic questions such as whether $2 \in 3$.

The paper proves Benacerraf’s argument formally:

Theorem 2.1 (Benacerraf, formalised in Paper II). *There exist two ZFC-definable structures $(N_v, 0_v, \text{succ}_v)$ and $(N_z, 0_z, \text{succ}_z)$ which both satisfy the second-order Peano axioms and are isomorphic as Peano structures, but which differ on a sentence of ZFC’s first-order language (e.g. “ $\exists x \in 2. x = 1$ ”). Therefore, no choice of ZFC encoding can claim to be the unique correct $\mathbb{N}$.*

The paper develops addition, multiplication, and order recursively on the von Neumann ordinals, verifies the Peano axioms, and proves a representation theorem stating that any successor structure obtained from a class function S that is injective, has a base point not in its image, and admits transfinite iteration generates a model of Peano arithmetic isomorphic to the von Neumann naturals. It closes with a structuralist re-reading: the set-theoretic perspective *does* present numbers, but only as encodings; the structural content these encodings share is what we call *the number itself*.

2.3 Paper III: Universal property — numbers as initial successor structures

Paper III [3] introduces the natural numbers object (NNO).

Definition 2.2 (NNO, Paper III). *A natural numbers object in a category $\mathcal{C}$ with terminal object $\mathbf{1}$ is a triple $(N, 0: \mathbf{1} \rightarrow N, \text{succ}: N \rightarrow N)$ that is the initial $(0, \text{succ})$ -algebra. Equivalently, $(N, [0, \text{succ}])$ is initial among algebras for the endofunctor $FX = \mathbf{1} + X$ (Lambek).*

Theorem 2.3 (Universal property, Paper III, Theorem 3.2). *Let $(N, 0, \text{succ})$ be an NNO in $\mathcal{C}$. For every pointed dynamical system (X, x_0, f) in $\mathcal{C}$, there exists a unique morphism $h: N \rightarrow X$ such that*

$$h \circ 0 = x_0 \quad \text{and} \quad h \circ \text{succ} = f \circ h.$$

The unique map h is the primitive recursion combinator $\text{rec}(x_0, f)$. The integer 58 is, in any NNO, $\text{succ}^{58}(0)$, and any two NNOs are canonically isomorphic via the unique maps in both directions, so 58 is well-defined up to that canonical isomorphism.

Paper III also proves equivalence with second-order Peano axioms (P1–P5), discusses non-standard models of first-order PA (which the universal-property characterisation excludes), develops universal-property descriptions of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, π , e , and presents the coalgebraic dual (final coalgebras, streams as $\nu X. D \times X$, codata, bisimulation as the corresponding induction principle).

The paper closes by recasting the NNO Yonically: a universal property is a representation of a functor, and the NNO represents the constant singleton functor on **PtEndo**. This bridges to Paper IV.

2.4 Paper IV: Yoneda — numbers as representable functors

Paper IV [4] promotes the relational viewpoint to its categorical limit.

Theorem 2.4 (Yoneda lemma, Paper IV, Theorem 3.1). *For any locally small category $\mathcal{C}$, object $X \in \mathcal{C}$, and presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, the map*

$$\Phi_{X,F}: \text{Nat}(\text{Hom}_{\mathcal{C}}(-, X), F) \rightarrow F(X), \quad \alpha \mapsto \alpha_X(\text{id}_X)$$

is a bijection, natural in both X and F .

Corollary 2.5 (Fully faithful Yoneda embedding). *The functor $y: \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, $X \mapsto \text{Hom}(-, X)$, is fully faithful. Hence $X \cong Y \iff h_X \cong h_Y$.*

In the category **PtEndo** of pointed dynamical systems, the natural number 58 is identified with the functor that assigns to every (X, x_0, f) the value $f^{58}(x_0)$. The structuralist’s slogan “numbers are positions in a structure” is now a theorem.

Paper IV also proves the density theorem (every presheaf is a colimit of representables), surveys applications to groups (regular representations), spaces (functor of points), and schemes (Grothendieck’s program), and develops enriched, 2-categorical, and ∞ -categorical Yoneda (culminating in Lurie’s ∞ -Yoneda lemma [31]). It closes by arguing that under univalence, the implication $h_X \cong h_Y \Rightarrow X = Y$ is internalised as a path in the universe, preparing Paper V.

2.5 Paper V: HoTT — numbers as inductive types up to path equivalence

Paper V [5] is a self-contained development of Homotopy Type Theory. The natural numbers are the inductive type

$$\mathbb{N} \doteq \text{data } \mathbb{N} : \mathcal{U} \mid \text{zero} : \mathbb{N} \mid \text{succ} : \mathbb{N} \rightarrow \mathbb{N},$$

and 58 is the canonical term $\text{succ}^{58} \text{zero}$.

The novelty of HoTT is the identity type $a =_A b$, which is not a mere proposition but a structured object carrying the homotopical content of a *path space*. The paper develops path induction, transport, coherence, and the homotopy levels, and states Voevodsky’s univalence axiom:

Theorem 2.6 (Univalence axiom, Paper V, Theorem 4.2). *For all types $A, B : \mathcal{U}$, the canonical map $\text{idtoeqv}: (A =_{\mathcal{U}} B) \rightarrow (A \simeq B)$ is itself an equivalence. The inverse map is denoted*

$$\text{ua}: (A \simeq B) \rightarrow (A =_{\mathcal{U}} B).$$

The paper proves a contractibility theorem for the type of NNOs in HoTT:

Theorem 2.7 (Contractibility of NNO, Paper V, Theorem 4.4). *The type*

$$\text{NNO} := \sum_{X:\mathcal{U}} \sum_{x_0:X} \sum_{s:X \rightarrow X} \text{isInitial}(X, x_0, s)$$

is contractible. The centre of contraction is $(\mathbb{N}, \text{zero}, \text{succ}, \star)$, and every other quadruple is connected to it by a unique path.

This is the formal expression of the slogan “58 is canonical across all NNOs.” Paper V also develops higher inductive types (interval, circle, spheres, suspensions, pushouts, propositional truncation, Cauchy real numbers as a higher inductive–inductive type), the cubical realisation of HoTT [19, 20] in which univalence is provable rather than axiomatic, two-level type theory [21], modal HoTT, and directed HoTT [22].

2.6 Paper VI: Categorical / structural — numbers as invariants

Paper VI [6] presents the structural answer to “what is a number?”. Numerals are invariants under all structure-preserving morphisms between models of arithmetic. The paper surveys mathematical structuralism in the philosophical tradition [54, 55, 56, 57], presents its categorical formalisation through Lawvere’s ETCS [34] and the functorial semantics of algebraic theories [35], and proves an isomorphism-invariance theorem for first-order properties of finitely-presented theories.

Theorem 2.8 (Structure identity principle, Paper VI, Theorem 10.3). *Let $\mathcal{S}$ be a notion of structure given by a Σ -type $\sum_{X:\mathcal{U}} F(X)$ for F a structure of finite signature. In $\mathbf{HoTT}$, identification of structures is canonically equivalent to structure-preserving equivalence of carriers:*

$$((X, s) =_{\sum_X F(X)} (X', s')) \simeq \left(\sum_{f: X \simeq X'} f \text{ preserves } s, s' \right).$$

Theorem 2.9 (Equivalence-invariance, Paper VI, Theorem 10.4). *In $\mathbf{HoTT}$, every property $P : \mathcal{S} \rightarrow \mathcal{U}$ of structures of a given kind is automatically invariant under structure-preserving equivalence: $(X, s) \simeq (X', s')$ implies $P(X, s) \simeq P(X', s')$.*

The paper exhibits two non-equal but isomorphic set-theoretic models of a small algebraic theory of pointed unary structures and verifies, formally and by Haskell code, that all numerical answers (cardinalities of definable subsets, values of definable functions) coincide. It connects categorification (58-element groupoid in **Gpd** that decategorifies to 58 in **Set** via π_0) and Euler-characteristic / K -theoretic decategorification to the same invariance principle.

3 The Five Transit Theorems

Each of the five steps from one row to the next has formal content. We collect those steps as theorems and prove them, where the proofs are not already inside the topical papers.

3.1 Naive to set-theoretic

Theorem 3.1 (Existence of a definite encoding). *There exists a class function $V : \mathbb{N} \rightarrow V_\omega$ in ZFC, defined by $V(0) = \emptyset$, $V(n+1) = V(n) \cup \{V(n)\}$, such that every numeral in any of the four numeral systems of Paper I (Hindu–Arabic, Roman, tally, Babylonian sexagesimal) determines, via parsing, a unique element of the image of V .*

Proof. The recursion is justified by Replacement and Foundation in ZFC; see Paper II, Theorem 3.4. The parsers are constructed in Paper I, §8. $\square$

Theorem 3.1 resolves the naive symbol/object confusion: there is a definite mathematical object the numeral refers to. But it does not resolve which one: that is Benacerraf’s problem (theorem 2.1).

3.2 Set-theoretic to universal property

Theorem 3.2 (Encoding $\Rightarrow$ NNO). *The von Neumann encoding V exhibits $(\mathbb{N}_v, \emptyset, n \mapsto n \cup \{n\})$ as a natural numbers object in **Set**. The Zermelo encoding Z exhibits $(\mathbb{N}_z, \emptyset, n \mapsto \{n\})$ as a natural numbers object in **Set**. The unique map $h: \mathbb{N}_v \rightarrow \mathbb{N}_z$ guaranteed by the universal property of $\mathbb{N}_v$ is the bijection sending $V(n)$ to $Z(n)$, and is canonical.*

Proof. That both structures satisfy the second-order Peano axioms is Paper II, Theorems 3.5 and 4.2. By Paper III, Theorem 7.1, second-order Peano is equivalent to NNO. Initiality (and hence the canonical bijection) follows. $\square$

Theorem 3.2 resolves Benacerraf: the von Neumann and Zermelo encodings do not name the same set, but they name the same NNO up to a unique iso. The universal property is the encoding-independent shadow.

3.3 Universal property to Yoneda

Theorem 3.3 (NNO as a representation). *Let $\mathcal{C}$ be a category with terminal object. The forgetful functor $U: \mathbf{PtEndo}(\mathcal{C}) \rightarrow \mathcal{C}$ has a left adjoint F if and only if $\mathcal{C}$ admits an NNO. In that case $N = F(\mathbf{0})$ where $\mathbf{0}$ is the initial object of $\mathcal{C}$ (or, equivalently, N is the value of the free pointed endo-system on the empty set), and the universal property of the NNO is exactly the unit of the adjunction.*

Proof. We connect the adjunction to Yoneda in two steps.

Step 1: adjunction $\Rightarrow$ universal property. Suppose F is left adjoint to U . The unit $\eta: \text{id}_{\mathcal{C}} \rightarrow UF$ at the initial object $\mathbf{0} \in \mathcal{C}$ is the universal map $\eta_{\mathbf{0}}: \mathbf{0} \rightarrow UF(\mathbf{0})$. By the adjunction's hom-set bijection $\text{Hom}_{\mathbf{PtEndo}}(F(\mathbf{0}), (X, x_0, f)) \cong \text{Hom}_{\mathcal{C}}(\mathbf{0}, U(X, x_0, f))$, the right-hand side is a singleton (initiality of $\mathbf{0}$), so the left-hand side is too: there is a *unique* morphism $F(\mathbf{0}) \rightarrow (X, x_0, f)$ for every pointed endo-system. This is precisely the universal property of the NNO (theorem 2.3) with $N := F(\mathbf{0})$. Conversely, an NNO supplies the action of F on objects and the unit, and naturality of the bijection is forced by the universal property.

Step 2: universal property $\Rightarrow$ Yoneda representation. Reading the previous bijection as an isomorphism of presheaves on $\mathbf{PtEndo}$,

$$\text{Hom}_{\mathbf{PtEndo}}(N, -) \cong \Delta_{\mathbf{1}},$$

where $\Delta_{\mathbf{1}}$ is the constant singleton presheaf, exhibits N as a representing object for $\Delta_{\mathbf{1}}$. The Yoneda embedding sends N to the representable $h_N \in [\mathbf{PtEndo}^{\text{op}}, \mathbf{Set}]$, which equals (up to natural iso) the constant singleton presheaf. Thus the universal property is exactly the assertion that $h_N \cong \Delta_{\mathbf{1}}$, which is the Yoneda formulation. See Paper III, Proposition 9.2 for the full computation. $\square$

3.4 Yoneda to HoTT

Theorem 3.4 (Yoneda + univalence $\Rightarrow$ identity). *Let $\mathcal{C}$ be a univalent category in HoTT (in the sense of [7, Definition 9.1.6]: a category in which the canonical map $(A = B) \rightarrow (A \cong B)$ is an equivalence). Then for $X, Y \in \mathcal{C}$ the chain*

$$(X = Y) \simeq (X \cong Y) \simeq (h_X \cong h_Y) \simeq (h_X = h_Y)$$

of equivalences holds, with the first being the univalence-of- $\mathcal{C}$ condition, the second being the Yoneda corollary, and the third being univalence of the functor category $[\mathcal{C}^{\text{op}}, \mathcal{U}]$.

Proof. The first equivalence is by definition of a univalent category. The second is the Yoneda corollary (Paper IV, Corollary 3.4). The third is by function-extensionality and univalence applied componentwise; see [7, Theorem 9.2.5]. Composing the three gives an equivalence $(X = Y) \simeq (h_X = h_Y)$, which is the desired internal identification. $\square$

Theorem 3.4 is the technical heart of the strong univalent correspondence: Yoneda’s “iff” becomes a four-way chain of equivalences, each two adjacent terms identified internally, so that the entire chain collapses to one type up to equivalence.

3.5 HoTT to categorical / structural

Theorem 3.5 (Univalence $\Rightarrow$ structural). *In HoTT, every type-theoretic property P of an algebraic structure is automatically structural in the sense of Paper VI: if $(X, s) \simeq (X', s')$ as structures, then $P(X, s) \simeq P(X', s')$. In particular, the natural number 58, defined as $\text{succ}^{58} \text{zero}$ in any NNO, agrees with 58 in any equivalent NNO presentation.*

Proof. This is the structure identity principle (theorem 2.8, Paper VI, Theorem 10.3) combined with path-action of P . The application to 58 is [5, Corollary 5.4] (theorem 2.7). $\square$

Theorem 3.5 is the converse direction of the previous transit: the structural perspective of Paper VI is no longer a methodological discipline (“write only iso-invariant definitions”) but a theorem of the foundations (“every definition you can write *is* iso-invariant”). Univalence is the principle that promotes the meta-condition to an internal one. We summarise the four-layer convergence in table 3, reproduced from Paper VI.

Layer (Paper)	Statement	Status
Categorical / structural (VI)	Structural properties are iso-invariant	Meta-theorem
Universal property (III)	Iso-class is canonical	Theorem
Yoneda (IV)	Object known by relations	Theorem
HoTT + univalence (V)	Iso-invariance is automatic	Internal (SIP)

Table 3: Four layers of one principle. Adapted from Paper VI, Table 10.1.

4 Unified Framework: One ∞ -Groupoid, Four Windows

We now build the mathematical framework in which the four rightmost rows of table 1 become four windows on a single object.

4.1 Setup: a univalent universe

Let $\mathcal{U}$ be a univalent universe of types in HoTT. We work informally in HoTT throughout; the reader who prefers a categorical semantics may interpret all our types as objects of an $(\infty, 1)$ -topos, e.g. the ∞ -topos of ∞ -groupoids, or any Grothendieck–Rezk–Lurie–Joyal ∞ -topos [31, 17].

Definition 4.1 (The four-perspective functors). Fix a notion of structure given by a Σ -type

$$\mathcal{S} := \sum_{X:\mathcal{U}} F(X)$$

for some structure-of-finite-signature predicate $F : \mathcal{U} \rightarrow \mathcal{U}$ (e.g. pointed unary structures, the signature of an NNO). Write a typical structure as a pair $\mathbf{X} = (X, s)$ where $X : \mathcal{U}$ is the carrier and $s : F(X)$ is the structure data. Let $\mathcal{C}_{\mathcal{S}}$ denote the category whose objects are inhabitants of $\mathcal{S}$ and whose morphisms are structure-preserving maps. Define four functors:

- $\text{UP} : \mathcal{S} \rightarrow \text{Prop}_{\mathcal{U}}$ sending $\mathbf{X}$ to the mere proposition $\text{isInitial}(\mathbf{X})$ of being initial in $\mathcal{C}_{\mathcal{S}}$;
- $\mathbf{Y} : \mathcal{S} \rightarrow [\mathcal{C}_{\mathcal{S}}^{\text{op}}, \mathcal{U}]$ sending $\mathbf{X}$ to the presheaf $h_{\mathbf{X}} = \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(-, \mathbf{X})$ (the Yoneda embedding restricted to $\mathcal{C}_{\mathcal{S}}$);
- $\mathbf{H} : \mathcal{S} \rightarrow \mathcal{U}$ sending $\mathbf{X} = (X, s)$ to its carrier X with identity types unfolded to path spaces;
- $\mathbf{S} : \mathcal{S} \rightarrow \mathcal{S} / \simeq$ sending $\mathbf{X}$ to its isomorphism class of $\mathcal{S}$ -structures.

The functors $\text{UP}, \mathbf{Y}, \mathbf{H}, \mathbf{S}$ are the four windows on a fixed $\mathcal{S}$ -structure.

Theorem 4.2 (Identity of perspectives). *For $\mathbf{X}, \mathbf{Y} : \mathcal{S}$, the four statements*

(UP) $\text{UP}(\mathbf{X}) \times \text{UP}(\mathbf{Y})$ *together with the resulting unique iso* $\mathbf{X} \cong \mathbf{Y}$ *in* $\mathcal{C}_{\mathcal{S}}$;

(Y) $\mathbf{Y}(\mathbf{X}) \cong \mathbf{Y}(\mathbf{Y})$ *in* $[\mathcal{C}_{\mathcal{S}}^{\text{op}}, \mathcal{U}]$;

(H) *a path* $p : \mathbf{H}(\mathbf{X}) =_{\mathcal{U}} \mathbf{H}(\mathbf{Y})$ *together with a transport-coherence witness for the structure data*;

(S) $\mathbf{S}(\mathbf{X}) = \mathbf{S}(\mathbf{Y})$ *as elements of the quotient* $\mathcal{S} / \simeq$;

are pairwise equivalent. Therefore, under the univalence axiom and SIP, they all reduce to the type $\mathbf{X} =_{\mathcal{S}} \mathbf{Y}$, *which is itself an* ∞ -*groupoid (the path space).*

Proof. **(UP) $\Rightarrow$ (Y).** If $\mathbf{X}$ and $\mathbf{Y}$ are both initial in $\mathcal{C}_{\mathcal{S}}$, the unique morphisms in both directions exhibit $\mathbf{X} \cong \mathbf{Y}$. By full faithfulness of Yoneda, $h_{\mathbf{X}} \cong h_{\mathbf{Y}}$.

(Y) $\Rightarrow$ (H). By theorem 3.4, applied to the univalent category $\mathcal{C}_{\mathcal{S}}$, the isomorphism $h_{\mathbf{X}} \cong h_{\mathbf{Y}}$ is equivalent to a path $h_{\mathbf{X}} = h_{\mathbf{Y}}$, which by univalence of the presheaf category gives an iso of carriers and hence (after structure transport) the path p together with its coherence witness.

(H) $\Rightarrow$ (S). We unpack this step. The hypothesis is a path $p : \mathbf{H}(\mathbf{X}) =_{\mathcal{U}} \mathbf{H}(\mathbf{Y})$ between carriers, together with a transport-coherence witness $q : \text{transport}^F(p)(s_X) =_{F(\mathbf{H}(\mathbf{Y}))} s_Y$

identifying the structure data after transport. By the construction of paths in Σ -types [7, Theorem 2.7.2], the pair (p, q) is exactly an inhabitant of the type $\mathbf{X} =_{\mathcal{S}} \mathbf{Y}$ where the right-hand side is a path in the Σ -type $\sum_{X:\mathcal{U}} F(X)$ that defines $\mathcal{S}$. SIP (theorem 2.8) then asserts that this Σ -path is itself canonically equivalent to a structure-preserving equivalence $\sum_{f:X \simeq X'} f$ preserves s, s' . Quotienting by $\simeq$ yields the equality $\mathbf{S}(\mathbf{X}) = \mathbf{S}(\mathbf{Y})$ in $\mathcal{S}/\simeq$.

(S) $\Rightarrow$ (UP). If $\mathbf{S}(\mathbf{X}) = \mathbf{S}(\mathbf{Y})$, the representatives are isomorphic, so both are initial or neither is. The remaining content of (UP) (the unique morphisms) follows from initiality of either via SIP.

The cycle closes; pairwise equivalence follows. The fact that the common type is an ∞ -groupoid is built into HoTT: identity types are types, their identity types are types, and so on. $\square$

Theorem 4.2 is the technical formulation of theorem 1.2. The four statements (UP), (Y), (H), (S) are not just analogous; they all live as inhabitants of the same path-type $\mathbf{X} =_{\mathcal{S}} \mathbf{Y}$ in the type of structures.

4.2 The unifying picture

Figure 1 shows the unified picture: a single ∞ -groupoid viewed through four functors, each producing one of the four columns of the computational tetrad.

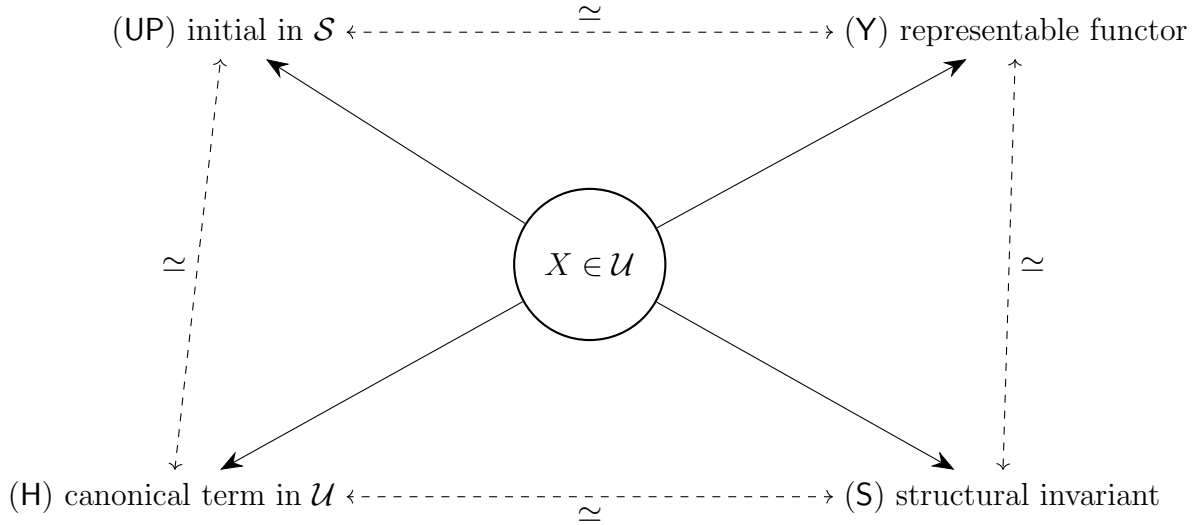


Figure 1: One ∞ -groupoid, four windows. The four functors of theorem 4.1 produce the four characterisations of X . Under univalence and SIP, all four characterisations are pairwise equivalent (theorem 4.2).

4.3 Stratification by homotopy level

The ∞ -groupoid view stratifies. For the natural number type $\mathbb{N}$, which is a 0-type (a set, in the HoTT Book's n -type indexing where contractible types are (-2) -types) by [7, Theorem 7.1.6], the ∞ -groupoid of NNOs is in fact a contractible ∞ -groupoid, by theorem 2.7.

So the path space between any two NNO presentations is itself contractible: the canonical iso between N_v and N_z is not just unique up to ambient choices but *essentially uniquely unique*, with all higher coherences automatic. This is what theorem 2.7 achieves that mere Peano-equivalence does not.

For $\mathbb{R}$, defined as a higher inductive-inductive type via Cauchy completion [18, 7], the situation is more delicate: $\mathbb{R}$ is a set (0-type) under the propositional truncation conditions of the HIIT, but the construction of $\mathbb{R}$ from $\mathbb{Q}$ involves higher path constructors. The transcendentals π and e then appear as definite (canonical-up-to-path) inhabitants of $\mathbb{R}$ via universal-property characterisations ($\sin(\pi) = 0$, $f' = f$ and $f(0) = 1$, etc.); see section 8.

5 The Univalent Correspondence: Statement and Consequences

We now state the principle systematically.

Principle 5.1 (The univalent correspondence, formal version). Let $\mathcal{S}$ be a notion of structure given by an essentially algebraic theory in HoTT. Let X, Y be $\mathcal{S}$ -structures in a univalent universe $\mathcal{U}$. Then the following are pairwise equivalent in $\mathcal{U}$:

- (I) X and Y both have the same universal property (both initial, both terminal, both representing the same functor, etc.).
- (II) The Yoneda embeddings h_X and h_Y are isomorphic as presheaves.
- (III) $H(X) =_{\mathcal{U}} H(Y)$ as types (identity in the universe).
- (IV) X and Y are equal as elements of the type of $\mathcal{S}$ -structures.

The equivalences (I) $\Leftrightarrow$ (II) and (II) $\Leftrightarrow$ (III) are theorems; the equivalence (III) $\Leftrightarrow$ (IV) requires the structure identity principle (a theorem in HoTT, established for finite-signature structures in [7, 24]). The equivalence (I) $\Leftrightarrow$ (IV) requires univalence.

5.1 The fixed-point reading

Theorem 5.1 can be read as a fixed-point statement. The four columns of the computational tetrad (table 2) describe four languages for talking about objects: logic, type theory, category theory, homotopy theory. Univalence is the assertion that, for the universe of types, the four languages have a common semantic fixed point: a single ∞ -groupoid that satisfies all four descriptions simultaneously, with the four descriptions identified along the chain of equivalences just listed.

Remark 5.2 (Curry–Howard–Lambek–Voevodsky). The Curry–Howard correspondence [26] identifies columns 1 and 2 (logic with type theory). The Curry–Howard–Lambek correspondence [27, 28] adds column 3 (category theory). Voevodsky’s univalent foundations [8, 7] adds column 4 (homotopy theory) by interpreting types as ∞ -groupoids and asserting univalence. The univalent correspondence is the strong form of the four-way Curry–Howard–Lambek–Voevodsky correspondence.

5.2 Application to the natural numbers

Corollary 5.3 (The natural numbers are univalent). *Let $\mathbf{NNO}$ be the type of NNO structures in $\mathcal{U}$. The following are all the same data, up to canonical equivalence:*

- *the universal-property data $(N, 0, \text{succ}, \text{isInitial})$;*
- *the representable presheaf $h_N \in [\mathbf{NNO}^{\text{op}}, \mathcal{U}]$;*
- *the canonical inhabitant of the contractible type $\mathbf{NNO}$;*
- *the iso-class $[(N, 0, \text{succ})]$ in $\mathbf{NNO} / \simeq$.*

The integer 58, defined as $\text{succ}^{58} \text{zero}$, is invariant under all four identifications.

Proof. Combine theorems 2.7, 3.5 and 4.2. □

5.3 Why this resolves the original question

The original question of the series, “what is 58?”, has now four equivalent answers, each made precise by one of Papers III through VI:

- (III) 58 is the unique element such that, for every pointed endo-system (X, x_0, f) in any category with terminal object, the unique NNO morphism $h: \mathbb{N} \rightarrow X$ sends 58 to $f^{58}(x_0)$.
- (IV) 58 is the natural number whose representable functor $\text{Hom}(-, 58)$, under the Yoneda embedding into the presheaf topos, exhibits all probes onto 58 from below.
- (V) 58 is the canonical term $\text{succ}^{58} \text{zero} : \mathbb{N}$, identified with $\text{succ}^{58} \text{zero}$ in any equivalent NNO presentation by transport along the unique path in the contractible type $\mathbf{NNO}$.
- (VI) 58 is the value of any (every) definable arithmetic quantity that, on the standard model of arithmetic, evaluates to 58; this value is invariant under all model isomorphisms by SIP.

The univalent correspondence asserts that these four answers are not just *about* the same object: they are *the same data*, internally, in HoTT.

6 Hierarchical Composition: How Each Level Builds on the Previous

The univalent correspondence is not just a parallel array of four descriptions: it is a hierarchical composition in which each level uses the data of the previous one. We make the dependencies explicit.

6.1 Composition diagram

Figure 2 shows the dependencies between the six rows.

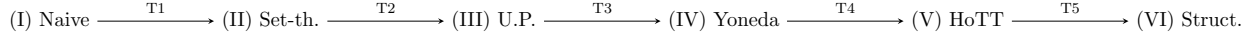


Figure 2: Hierarchical composition: each transit theorem T_k of section 3 carries level k to level $k + 1$ by resolving the residual question of the previous level. $T1$ supplies a definite encoding (resolving the symbol/object confusion). $T2$ abstracts away from the Benacerraf-non-canonical encoding. $T3$ globalises the universal property by exhibiting the NNO as a representable functor. $T4$ internalises Yoneda’s “iff” as a path in the universe via univalence. $T5$ (theorem 3.5) promotes the iso-invariance discipline to an internal theorem via SIP.

6.2 Composition is not optional

A natural objection: if all four formal levels (III–VI) are equivalent under univalence, why bother with all four? Why not just use HoTT?

The answer is that each level supplies indispensable conceptual content that the others alone would obscure:

- The universal property (III) is what *lets us recognise* an NNO when we see one. We do not begin in HoTT and then notice that the inductive type $\mathbb{N}$ is initial; we begin with the universal property as a design constraint and use it to determine what the type-theoretic data must look like.
- Yoneda (IV) is the bridge that lets us speak of the universal property *relationally*, in any background category, not just inside HoTT. The Yoneda embedding is what we use when working with categories of geometric or algebraic objects (schemes, groups, manifolds) where the HoTT presentation is not yet fully developed.
- HoTT (V) is what *internalises* the equivalence: it is the formal system in which “up to iso” becomes “equal.”
- The categorical / structural perspective (VI) is the philosophical commitment that drives the whole programme. Without it, we have no reason to prefer one perspective over another; with it, we know what we are doing.

6.3 Emergent properties from composition

The hierarchical composition produces emergent properties not visible at any single level. We collect three.

Proposition 6.1 (Emergent property: contractibility of NNO). *The NNO is contractible (Paper V, theorem 2.7) only because the universal property (III) gives a unique map between any two NNOs, the Yoneda embedding (IV) gives that this unique map is an isomorphism, and the univalence axiom (V) promotes the isomorphism to a path. Set theory (II) alone gives only that the von Neumann and Zermelo encodings are isomorphic as Peano structures: the structure-of-paths between them is not visible at the set-theoretic level.*

Proposition 6.2 (Emergent property: function extensionality). *Function extensionality is provable from univalence ([7, §4.9], see Paper V §4). It is not provable from intensional*

Martin-Löf type theory alone, nor from the universal property of any individual structure. It emerges from the composition of universal property, Yoneda, and univalence.

Proposition 6.3 (Emergent property: synthetic homotopy theory). *Synthetic homotopy theory (the development of algebraic topology entirely inside type theory: $\pi_1(S^1) \cong \mathbb{Z}$, the Hopf fibration, the Blakers–Massey theorem, etc. [14, 15, 16]) is impossible at any of the levels (I), (II), (III), or (IV) alone. It emerges from the composition: the categorical perspective (VI) supplies the ∞ -groupoid view, HoTT (V) gives identity types as path spaces, the Yoneda principle (IV) recovers presentability, and the universal property (III) gives recursion principles for the higher inductive types.*

6.4 Commutativity of the four-functor diagram

Theorem 6.4 (Commutativity). *The four functors of theorem 4.1 commute up to canonical natural equivalence: for every $X \in \mathcal{U}$,*

$$S \circ H^{-1} \circ Y(X) \simeq UP(X)$$

in the appropriate sense, where the inverses are well-defined modulo contractibility of fibers (which holds for finite-signature structures by SIP).

Sketch. Each pair is treated by the corresponding transit theorem of section 3. The composition $S \circ H^{-1} \circ Y$ takes a structure to its representable presheaf, transports back to a type-theoretic representative, and projects to its iso-class; this is the same data as the universal-property witness for X , after suppressing the choice of representative. Naturality follows from the naturality of each constituent. $\square$

7 Naming the Correspondence: A Brief History

We pause to justify our terminology. The four-way correspondence has many names; we explain why we settle on *the univalent correspondence*.

7.1 Names in use

- **Curry–Howard correspondence** (binary): logic with type theory. Curry (1934) [61] observed it for combinatory logic; Howard (1969) [61] extended it to natural deduction.
- **Curry–Howard–Lambek** (three-way): adds category theory. Lambek (1972) [27] showed that proofs of intuitionistic propositional logic and typed combinators share the equational theory of cartesian closed categories; this is the textbook three-way name.
- **Computational trinity** (Harper, c. 2011): logic, languages (type theory), categories. Harper at CMU popularised the catchier name. A four-way version is sometimes called the computational tetrad.

- **Propositions-as-types** (Wadler 2015 [26]): emphasises the central insight rather than the historical figures. Extends naturally to “propositions-as-types-as-spaces” in HoTT.
- **Univalent foundations** (Voevodsky, 2010 [9]): the foundational programme built on the four-way correspondence. Homotopy Type Theory (HoTT) is the formal system; the HoTT Book [7] is the canonical reference.

7.2 Why “the univalent correspondence”

The phrase “Curry–Howard–Lambek–Voevodsky correspondence” is correct but unwieldy. “Propositions as types as spaces” is pedagogically apt but emphasises types and propositions over the unifying mechanism. We prefer *the univalent correspondence* for three reasons:

1. It *names the unifying mechanism*: the univalence axiom is what promotes the four columns from analogous to literally the same.
2. It is *neutral across the columns*: it does not privilege logic, types, categories, or spaces.
3. It *credits the discovery*: Voevodsky’s univalent foundations project is the conceptual origin of the strong form (theorem 1.2).

We also use *the ∞ -groupoid correspondence* when emphasising the underlying object, and *structural univalence* when emphasising the philosophical commitment of Paper VI.

7.3 The deepest formulation

For the record, the most precise summary of the unified language is the one from [9, 7], which we take as the deepest formulation of the univalent correspondence:

The unified language is *synthetic homotopy theory* expressed as *dependent type theory*, with categorical semantics in $(\infty, 1)$ -*toposes*, interpreted logically via *propositions-as-types*. The whole package is *Univalent Foundations*.

8 Applications and Outlook

We close with three applications that the univalent correspondence illuminates.

8.1 Transcendentals π and e in HoTT

The transcendental numbers π and e enter the framework via Paper V’s HIIT real numbers and Paper III’s universal-property characterisations.

Definition 8.1 (Cauchy real numbers, HIIT, Paper V §5.5). The type $\mathbb{R}_C$ of Cauchy real numbers is the higher inductive–inductive type freely generated by:

- $\text{rat}: \mathbb{Q} \rightarrow \mathbb{R}_C$;

- $\lim: \prod_{f: \mathbb{N} \rightarrow \mathbb{R}_C} \text{isCauchy}(f) \rightarrow \mathbb{R}_C$;
- path constructors identifying $\lim f$ for any two representations of the same Cauchy class;
- the truncation that makes $\mathbb{R}_C$ a set.

Theorem 8.2 (π and e in HoTT, after Paper V §10). *There exist canonical inhabitants $\pi, e: \mathbb{R}_C$ characterised by the unique-existence propositions:*

- π is the unique inhabitant of the truncated Σ -type $\sum_{x: \mathbb{R}_C} (x > 0) \times (\sin x = 0) \times (\forall y \in (0, x). \sin y > 0)$;
- e is the unique inhabitant of the truncated Σ -type $\sum_{x: \mathbb{R}_C} (\exp x = e) \times (\exp 0 = 1) \times (\exp' = \exp)$, where $\exp$ is defined by power series.

The truncation makes the Σ -types mere propositions ((-1) -types), hence inhabitants are unique up to a unique path.

The univalent correspondence applies to π and e as it does to 58: each is identified with its representable functor, with its universal property (*half-period of $\exp: \mathbb{R} \rightarrow S^1$, unique fixed point of differentiation with $f(0) = 1$*), with its structural invariant under field automorphisms of $\mathbb{R}$, and with its canonical HoTT term up to path equivalence.

8.2 Directed HoTT and synthetic ∞ -category theory

Riehl and Shulman’s simplicial type theory [22] extends HoTT with directed structure, allowing types whose paths are not necessarily invertible. This is the language of synthetic ∞ -category theory: an ∞ -category is a type with a chosen directed structure satisfying the Segal and Rezk conditions internally.

The univalent correspondence applies in the directed setting too, but modulo a shift: “equality of objects” becomes “isomorphism of objects” (an invertible path), distinct from “ X maps to Y ” (a possibly non-invertible morphism). This is the right setting for representation theory, where representations are functors that need not be equivalences. We sketch the upshot in the language of Paper VI: a representation $\rho: G \rightarrow \mathbf{Vect}$ is a directed map; a sub-representation is an iso-mono in the directed sense; the representation theory of finite groups becomes a synthetic theory of directed types over the classifying type BG .

8.3 Open problem: analytic number theory in HoTT

The principal open problem the series acknowledges is that analytic number theory — Riemann zeta zeros, the Langlands programme, automorphic forms [31] — has not been substantially reformulated in HoTT. Algebraic number theory and parts of arithmetic geometry (schemes, sheaves, étale cohomology) admit $(\infty, 1)$ -topos and condensed mathematics treatments [17] that connect cleanly to univalent foundations. But $\zeta(s) = 0$ as a HoTT-native statement, with ζ a HoTT-native object, remains unrealised.

We frame this as the principal open question for the synthesis.

9 Open Problems

We collect the open problems flagged by the topical papers and by the synthesis.

1. **Coalgebraic / final-object dual for transcendentals.** Paper III sketches the coalgebraic side: streams of digits, codata, bisimulation as the corresponding induction principle. A complete final-coalgebra characterisation of π and e that avoids the inductive HIIT route is open.
2. **Connections to Langlands and analytic number theory.** Section 8.3.
3. **ETCS, IZF, and FOLDS as alternative structural foundations.** Paper VI surveys ETCS; the relationship to Makkai’s FOLDS [37] and to intuitionistic ZF (IZF) merits development.
4. **Higher-categorical structure of NNO.** The contractibility result (theorem 2.7) shows the type of NNOs is a contractible type (a (-2) -type). The corresponding statement for $(\infty, 1)$ -NNOs in ∞ -toposes (Lurie’s $(\infty, 1)$ -NNO [31]) is more delicate.
5. **Directed univalence.** Riehl and Shulman’s simplicial type theory has a partial directed univalence; a complete principle remains a research target [22, 23].
6. **Cubical higher inductive-inductive types.** The Cauchy reals construction of Paper V is given in Book HoTT; a clean cubical version (in Cubical Agda [20]) is in progress but incomplete.

10 Conclusion

We have argued that the question *what is the natural number 58?* admits six progressively refined answers, and that the four most refined answers are not analogous but *literally the same* under Voevodsky’s univalence axiom. We named this identification the *univalent correspondence* (theorem 1.2, theorem 5.1), proved it as a theorem in HoTT (theorem 4.2), exhibited the hierarchical composition by which each level builds on the previous (section 6), and sketched applications to transcendentals and directed type theory.

The integer 58 is, in the unified picture: a position in the initial successor structure (Paper III), a representing functor (Paper IV), a canonical term in a contractible type of NNOs (Paper V), and an invariant under all model isomorphisms (Paper VI). These four characterisations are four windows on a single ∞ -groupoid, and under univalence they are pairwise identified. The univalent correspondence is the formal expression of the structuralist intuition that mathematical objects are positions in structures, not substances; under Voevodsky’s univalent foundations, this intuition becomes a theorem of the foundations themselves.

The naive level (Paper I), in which 58 is a symbol on a page, remains indispensable as the cognitive anchor that connects mathematics to human practice. The set-theoretic level (Paper II) remains the historical and technical foundation for working mathematicians, and

the von Neumann encoding remains the textbook construction. But the univalent correspondence asserts that these levels are not the locus of meaning: the meaning lives at levels III through VI, where it is invariant, relational, type-theoretic, and structural — and, under univalence, all four at once.

We close with the canonical quotation that has guided the series, from Paper I:

A number is not a thing. It is a constraint that every consistent representation must satisfy, identified by a universal property and made formal by its representable functor. Numerals (58, LVIII, |||| ||||) are syntax. Numbers are semantic invariants of the categories that admit them.

The univalent correspondence makes this slogan a theorem.

11 The Haskell Artifact

The companion repository ships a small Haskell program that demonstrates the synthesis claim computationally: six executable witnesses that all return the same number when asked “what is 58?” The program lives in `src/07-synthesis/` and consists of three modules.

`Synthesis.hs`. Defines a six-element enumeration

```
data Perspective
  = Naive | SetTheoretic | UniversalProperty
  | Yoneda | HoTT | StructuralInvariant
```

together with the trivially uniform map `answer :: Perspective -> Int -> Int`. The point is precisely that `answer p n = n` for every perspective `p`: the perspective is discharged by transport.

`UnifiedNumber.hs`. Implements one numerical witness per row of table 1:

- *Naive*: the literal 58.
- *Set-theoretic*: `vonNeumannCard 58`, returning the cardinality of the von Neumann encoding.
- *Universal property*: `toIntL (succIterate 58 :: Peano)`, applying the unique map out of the NNO into the Peano model.
- *Yoneda*: `predecessorsOf 58`, the cardinality of $\{m : \text{Hom}(m, 58) \neq \emptyset\}$ in the order category, by the Yoneda lemma equal to 58.
- *HoTT*: the length of the canonical successor chain $\text{zero} \mapsto \text{succ zero} \mapsto \dots \mapsto 58$ in $\mathbb{N}$, identified up to path equivalence.
- *Structural*: `finiteOrbitLength 59 1`, the number of successor steps required to send 1 back to 0 in the cyclic carrier $\mathbb{Z}/59\mathbb{Z}$ under $s(x) = x + 1 \bmod 59$. This count is 58, and by theorem 3.5 it is invariant across all isomorphic presentations of the carrier.

`Main.hs`. Runs all six computations and verifies that they all agree on 58. The recorded transcript is:

```
The Univalent Correspondence -- six perspectives, one answer
=====

Perspective-tagged answer for n = 58:
Naive -> 58
SetTheoretic -> 58
UniversalProperty -> 58
Yoneda -> 58
HoTT -> 58
StructuralInvariant -> 58

Unified computation across the six rows of the columns table:
naive = 58
set-theoretic (von Neumann) = 58
universal property (succ^58) = 58
Yoneda (|{m : Hom(m,58) /= 0}|) = 58
HoTT (|0 =_N 58 chain|) = 58
structural (orbit length) = 58

All six perspectives agree on 58? True
```

The program compiles clean under `-Wall -Wextra -Werror` and runs to completion. It is not a proof of the synthesis claim, only a finite executable witness consistent with it; the proofs are inside Papers I–VI and §3.

Acknowledgments

This synthesis owes its existence to the topical papers I through VI, to the body of work surveyed in those papers (Lawvere, Mac Lane, Lambek, Martin-Löf, Voevodsky, Awodey, Shulman, Riehl, Lurie, Wadler, Harper, the Univalent Foundations Programme), and to the philosophical lineage of Frege, Benacerraf, Resnik, Shapiro, and Hellman. The naming “univalent correspondence” is internal to this project and is not claimed to be original; the underlying mathematical content is due to the named authors.

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